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Cichon’s Conjecture on the Slow Growing Hierarchy

The unexpected power of a pointwise hierarchy

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https://tcs-informatik.uibk.ac.at/

Summary of Last Lecture

Concepts and Theorems

- terms and positions
- term rewrite systems
- (universal) termination of TRSs
- simplification orders
  - LPO
  - MPO
  - KBO
- well-founded monotone algebras

Theorem

- *$>_{\mathcal{A}}$ is reduction order for every WFMA $(\mathcal{A}, >)$*
- *TRS is terminating iff compatible with WFMA*

Simple Termination

(signatures are finite)

Definitions

- binary relation $>$ on terms is simplification order if
  1. closed under contexts
  2. closed under substitutions
  3. proper order
  4. subterm property $\quad u[t]_p > t$ when $p \neq \epsilon$
- TRS is simply terminating if compatible with simplification order
- TRS $\mathcal{E}\mathrm{mb}$ consists of all rewrite rules

$$\mathsf{f}(x_1, \ldots, x_n) \to x_i$$

- $\rhd_{\mathrm{emb}} \; = \; \to^*_{\mathcal{E}\mathrm{mb}}$ embedding

Theorem

simplification orders are well-founded

Proof Idea.

proof is based on Kruskal's Tree Theorem ∎

Theorem

simply terminating TRSs are terminating

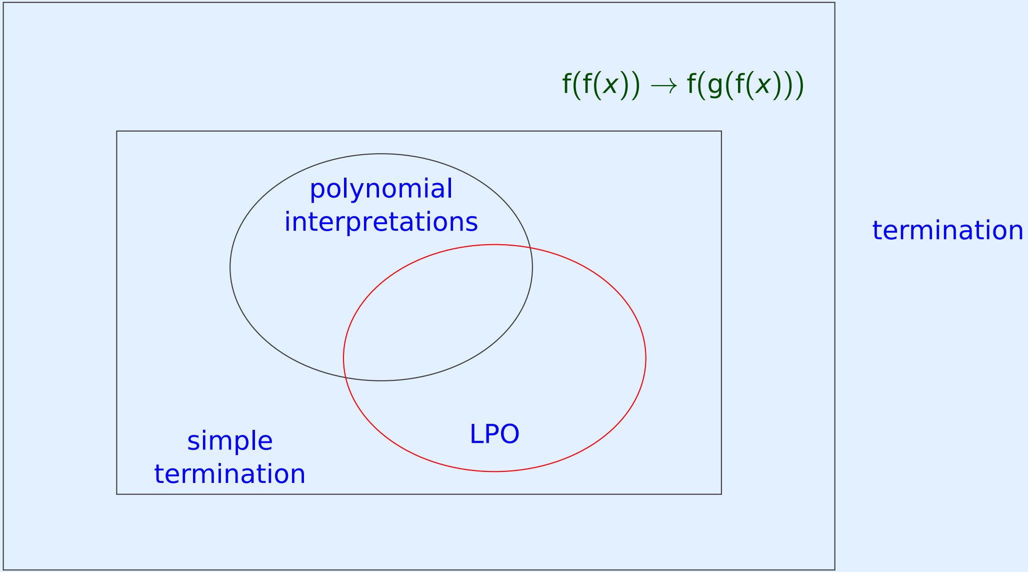
Lemma

TRS $\mathcal{R}$ is simply terminating iff $\mathcal{R} \cup \mathcal{E}\mathsf{mb}$ is terminating

Example

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \to_{\mathcal{E}\mathsf{mb}} \mathsf{f}(\mathsf{f}(x))$$

Picturing Simple Termination



A Brief History of Termination Methods

• interpretation method	Turing	1949
• polynomial interpretations	Lankford	1975
	Ben Cherifa, Lescanne	1987
• lexicographic path order	Schütte	1960
	Dershowitz	1982
	Kamin, Lévy	1980
• Knuth-Bendix order	Knuth, Bendix	1970
	Dick, Kalmus, Martin	1990
• recursive decomposition order		
	Jouannaud, Lescanne, Reinig	1982

Remark

traditional termination orders yield simple termination; modern techniques surpass this significantly

The Slow-Growing Hierarchy and Friends

Content

- derivational complexity
- complexities induced by simplification order
- complexities induced modern termination techniques
- Hydra Battle by Kirby and Paris
- subrecursive hierarchies

Definition

the derivation height of a term t wrt to T and $\to$

$$\mathrm{dh}(t, \to) = \max\{n \mid \exists u\ t \to^n u\}$$

$$\mathrm{dh}(n, T, \to) = \max\{\mathrm{dh}(t, \to) \mid \exists t \in T \text{ and } |t| \leqslant n\}$$

Definition (Derivational Complexity)

the derivational complexity with respect to a TRS $\mathcal{R}$

$$\mathrm{dc}_{\mathcal{R}}(n) = \mathrm{dh}(n, \text{"all terms"}, \to_{\mathcal{R}})$$

Example

consider TRS $\mathcal{R}_1$

$$\mathsf{ack}(0, y) \to \mathsf{S}(y) \qquad \mathsf{ack}(\mathsf{S}(x), \mathsf{S}(y)) \to \mathsf{ack}(x, \mathsf{ack}(\mathsf{S}(x), y))$$

$$\mathsf{ack}(\mathsf{S}(x), 0) \to \mathsf{ack}(x, \mathsf{S}(0))$$

$\mathrm{dc}_{\mathcal{R}_1}$ majorises every primitive recursive function

How To Analyse Complexity

$$t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \dots \to_{\mathcal{R}} t_n$$

consider

1. $\exists$ termination technique such that
2. termination of $\mathcal{R}$ is certified

Fact

termination techniques can be used to measure the derivation height

Example

polynomial interpretations induce double exponential derivational complexity

D. Hofbauer and C. Lautemann.
Termination Proofs and the Length of Derivations.
In *Proc. 3rd RTA*, pages 167–177, 1989.

Results

- introduction of derivation height, derivational complexity
- derivational complexity as measure of a termination technique

Theorem

polynomial interpretations induce double-exponential derivational complexity

Lemma 1

$\forall\, \mathcal{R}$ terminating via a polynomial interpretation
$\exists\, c \in \mathbb{R}, c > 0\ \forall$ terms s: $\mathrm{dh}(s, \to_{\mathcal{R}}) \leqslant 2^{2^{c \cdot |s|}}$

Lemma 2

$\exists\, \mathcal{R}$ terminating via a polynomial interpretation
$\exists\, c \in \mathbb{R}$, $c > 0$ for infinitely many terms t: $\mathsf{dh}(t, \to_{\mathcal{R}}) \geqslant 2^{2^{c \cdot |t|}}$

Proof of Lemma 2

consider $\mathcal{R}_3$:

$$x + 0 \to x \qquad \mathsf{d}(0) \to 0 \qquad \mathsf{d}(\mathsf{S}(x)) \to \mathsf{S}(\mathsf{S}(\mathsf{d}(x)))$$
$$x + \mathsf{S}(y) \to \mathsf{S}(x + y) \qquad \mathsf{q}(0) \to 0 \qquad \mathsf{q}(\mathsf{S}(x)) \to \mathsf{q}(x) + \mathsf{S}(\mathsf{d}(x))$$

together with the polynomial interpretation $\mathcal{A}$

$$0_{\mathcal{A}} = 2 \qquad \mathsf{S}_{\mathcal{A}}(n) = n + 1 \qquad n +_{\mathcal{A}} m = n + 2m$$
$$\mathsf{d}_{\mathcal{A}}(n) = 3n \qquad \mathsf{q}_{\mathcal{A}}(n) = n^3$$

- S defines the successor function
- d defines the doubling function, i.e., $\mathsf{d}(\mathsf{S}^n(0)) \xrightarrow{*} \mathsf{S}^{2n}(0)$
- q defines the square function, i.e., $\mathsf{q}(\mathsf{S}^n(0)) \xrightarrow{*} \mathsf{S}^{n^2}(0)$

Proof (cont'd)

from this we get:

$$t_m = \mathsf{q}^{m+1}(\mathsf{S}^2(0)) \xrightarrow{*} \mathsf{q}(\mathsf{S}^{2^{2^m}}(0)) \xrightarrow{\geqslant 2^{2^m}} \mathsf{S}^{2^{2^{m+1}}}(0)$$

we conclude, for all $m \geqslant 1$

$$\mathsf{dh}(t_m, \to_{\mathcal{R}_3}) \geqslant 2^{2^m} = 2^{2^{|t_m|-4}} \geqslant 2^{2^{c \cdot |t_m|}}$$

where $c \leqslant \frac{1}{5}$ ■

Question 1

Is $\mathcal{R}_1$, ie. the TRS implementing the Ackermann function, polynomially terminating?

Answer

- the TRS $\mathcal{R}_1$ implements the Ackermann function, that grows much faster than double-exponentially
- hence, the answer is (very much) no!

Reduction Orders Induce Derivational Complexities

Theorem

the multiset path order induces primitive recursive derivational complexity; this bound is tight

Theorem

the lexicographic path orders induce multiple recursive derivational complexity; this bound is tight[1]

D. Hofbauer.
Termination proofs by multiset path orderings imply primitive recursive derivation lengths.
TCS, 105:129–140, 1992.

A. Weiermann.
Complexity bounds for some finite forms of Kruskal's theorem. JSC, 18(5):463–488, November 1994.

[1] The multiple recursive functions are based on Rozsa Peter's generalisations of the Ackermann function.

Theorem

the Knuth-Bendix orders induce derivational complexities that are contained in the Ackermann function, more precisely, $\mathsf{dc}_{\mathcal{R}}(n) \in \mathsf{Ack}(\mathsf{O}(n), 0)$, *whenever* $\mathcal{R} \subseteq >_{\mathsf{kbo}}$*; this bound is tight*

Remark

- derivational complexity of modern termination techniques analysed, including modular techniques (like the dependency pair method) or methods not based on reduction orders (like match-bound techniques)
- for (almost) all automated techniques, the complexity induces is bounded by a multiple recursive function

I. Lepper.
Derivation lengths and order types of Knuth-Bendix orders. TCS, 269(1-2):433–450, 2001.

A. Schnabl.
University of Innsbruck. PhD thesis, Derivational Complexity Analysis revisited, 2012.

Back to Proof Theory

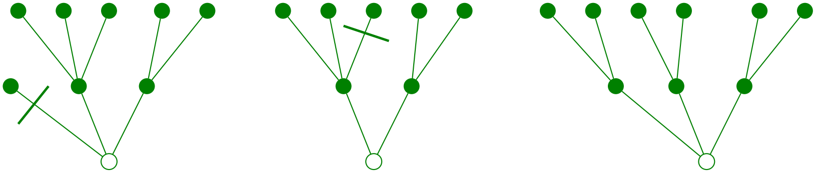
The Hydra Battle by Kirby and Paris

1. the beast is a finite tree, each leaf corresponds to a head; Hercules chops off heads of the Hydra, but the Hydra regrows:
   - if the cut head has a pre-predecessor, then the remaining subtree issued from this node is multiplied by the stage of the game.
   - otherwise the Hydra ignores the loss.
2. Hercules wins, when the beast is reduced to the empty tree.

Theorem

termination of the battle is independent: PA $\nvdash$ the battle terminates

Example



$(H_1, 1)$ $(H_2, 2)$ $(H_3, 3)$

Definition

a strategy is a mapping determining which head Hercules chops off at each stage

Theorem (Kirby, Paris)

Every strategy is a winning strategy.

Proof.

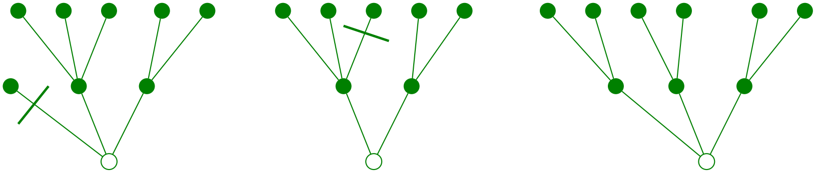
In proof, associate with each Hydra an ordinal $< \epsilon_0$:

- To each leaf assign 0.
- To each other node *v* assign $\omega^{\alpha_1} \oplus \cdots \oplus \omega^{\alpha_n}$, if α_i are the ordinals assigned to the successors of *v*.
- The ordinal representing the Hydra, is the ordinal assigned to the root.

$\oplus$ denotes the natural sum.

∎

Example



$(\omega^3 \oplus \omega^2 \oplus 1, 1) \qquad (\omega^3 \oplus \omega^2, 2) \qquad (\omega^2 \cdot 3, 3)$

$\omega^3 + \omega^2 + 1 \quad > \quad \omega^3 + \omega^2 \quad > \quad \omega^2 \cdot 3$

The Standard Hydra Battle

- define a specific recursive strategy
- associate with $\alpha \in$ Cantor Normal Form, $\alpha_n \in$ Cantor Normal Form:

$$\alpha_n = \begin{cases} 0 & \text{if } \alpha = 0 \\ \beta & \text{if } \alpha = \beta + 1 \\ \beta + \omega^\gamma \cdot n & \text{if } \alpha = \beta + \omega^{\gamma+1} \\ \beta + \omega^{\gamma_n} & \text{if } \alpha = \beta + \omega^\gamma \text{ and } \gamma \in \text{Lim} \end{cases}$$

Definition (The Standard Hydra Battle)

a Hydra is an ordinal in CNF; the Hydra battle is a sequence of configurations (α, n):

$$(\alpha, n) \underbrace{\Longrightarrow}_{\text{one step}} (\alpha_n, n+1)$$

Subrecursive Hierarchies

Definition

similar to the (standard) Hydra battle, we define the family of fundamental sequences $\lambda[x]_{x\in\mathbb{N}}$ as follows (λ limit ordinal):

$$\lambda[x] = \begin{cases} x+1 & \text{if } \lambda = \omega \\ \beta + \omega^{\alpha}\cdot(x+1) & \text{if } \lambda = \beta + \omega^{\alpha+1} \\ \beta + \omega^{\alpha[x]} & \text{if } \lambda = \beta + \omega^{\alpha}, \alpha \text{ limit} \end{cases}$$

NB: $\sup\{\lambda[x] \mid x \in \mathbb{N}\} = \lambda$

Definition

the Hardy functions $(\mathsf{H}_{\alpha})_{\alpha\in\mathcal{O}}$ are defined as follows:

$$\mathsf{H}_0(x) = x \qquad \mathsf{H}_{\alpha+1}(x) = \mathsf{H}_{\alpha}(x+1) \qquad \mathsf{H}_{\lambda}(x) = \mathsf{H}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

- the length of the (standard) Hydra battle is (almost) directly expressible via the Hardy function
- as observed by Lepper: “Hydra” and “Hardy” are anagrams ☺

Definition

the family of slow-growing functions $(G_\alpha)_{\alpha \in \mathcal{O}}$ is defined as follows:

$$G_0(x) = 0 \qquad G_{\alpha+1}(x) = G_\alpha(x) + 1 \qquad G_\lambda(x) = G_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Example

$$G_\omega(x) = x + 1 \qquad G_{\omega^{\omega^\omega}}(x) = (x+1)^{x+1^{x+1}} \qquad G_{\omega\cdot 2}(10) = (10+1)\cdot 2 = 22$$

the families $(\mathrm{G}_\alpha)_{\alpha\in\mathcal{O}}$, $(\mathrm{H}_\alpha)_{\alpha\in\mathcal{O}}$ form proper hierarchies, that is, for $\alpha > \beta$:

$$\exists\, c \text{ such that } \forall\, x \geqslant c\colon \mathrm{G}_\alpha(x) > \mathrm{G}_\beta(x),\ \mathrm{H}_\alpha(x) > \mathrm{H}_\beta(x),$$

Example

$$\forall x \geqslant 1 \qquad \mathrm{G}_{\omega^\omega}(x) = (x+1)^{x+1} > (x+1) = \mathrm{G}_\omega(x)$$

$$\forall x \geqslant 1 \qquad \mathrm{G}_\omega(x) = x+1 \not> y = \mathrm{G}_y(x) \qquad \text{whenever } y > x$$

Theorem

Simplification orders induce functions elementary in H_Λ, where Λ denotes the "small Veblen ordinal" (sometimes denoted as $\theta(\Omega^\omega)$), where Ω stands either for the first uncountable ordinal); note that $\Lambda \ggg \epsilon_0$.

I. Lepper.
Simply terminating rewrite systems with long derivations.
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Further Reading

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Thank You for Your Attention!