universität innsbruck

Cichon’s Conjecture on the Slow Growing Hierarchy

The unexpected power of a pointwise hierarchy

Georg Moser

https://tcs-informatik.uibk.ac.at/

Summary of Last Lecture

Concepts and Theorems

- derivational complexity
- reduction orders induce derivational complexities
- Hydra battle and it's independence
- subrecursive hierarchies

Theorem

the lexicographic path orders induce multiple recursive derivational complexity; this bound is tight

Theorem

the Knuth-Bendix orders induce derivational complexities that are contained in the Ackermann function, more precisely, $\mathsf{dc}_{\mathcal{R}}(n) \in \mathsf{Ack}(\mathsf{O}(n), 0)$*, whenever* $\mathcal{R} \subseteq >_{\mathsf{kbo}}$*; this bound is tight*

Cichon’s Conjecture and Counterexample

Content

- Girard’s hierarchy comparison theorem
- Cichon’s conjecture and it’s counterexample
- Buchholz’s proof of Weiermann’s result
- What makes a pointwise hierarchy slow growing?
- Hydra battle in rewriting
- Wrap Up

Girard's Hierarchy Comparison Theorem

recall that $(\mathsf{G}_\alpha)_{\alpha \in \mathcal{O}}$ is defined as follows:

$$\mathsf{G}_0(x) = 0 \qquad \mathsf{G}_{\alpha+1}(x) = \mathsf{G}_\alpha(x) + 1 \qquad \mathsf{G}_\lambda(x) = \mathsf{G}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

and that the Hardy functions $(\mathsf{H}_\alpha)_{\alpha \in \mathcal{O}}$ are defined as follows:

$$\mathsf{H}_0(x) = x \qquad \mathsf{H}_{\alpha+1}(x) = \mathsf{H}_\alpha(x+1) \qquad \mathsf{H}_\lambda(x) = \mathsf{H}_{\lambda[x]}(x) \quad (\lambda \text{ limit})$$

Theorem (Girard, Fairtlough and Wainer)

Let $E(\cdot)$ denote closure under elementary functions, then we have

$$\bigcup_{\alpha < \text{"Howard-Bachmann ordinal"}} E(\mathsf{G}_\alpha) \approx \bigcup_{\alpha < \epsilon_0} E(\mathsf{H}_\alpha)$$

NB. $\bigcup_{\alpha < \epsilon_0} E(\mathsf{H}_\alpha)$ characterises the provable recursive functions of Peano Arithmetic

Cichon's Conjecture

The derivational complexity induced by any termination order of order type α is bounded by the slow-growing hierarchy indexed by α

Example

- consider KBO and the derivational complexity induced
- the order type of KBO is ω^ω
- on the other hand $\mathsf{dc}_{\mathcal{R}}(n) \in \mathsf{Ack}(\mathsf{O}(n), 0)$, whenever $\mathcal{R} \subseteq >_{\mathsf{kbo}}$ and the bound is tight
- recall that $\mathsf{G}_{\omega^\omega}(x) = (x + 1)^{x+1}$

D. Hofbauer.
Termination proofs and derivation lengths in term rewriting systems. PhD thesis, Technical University of Berlin, 1992.

I. Lepper.
Derivation lengths and order types of Knuth-Bendix orders. TCS, 269(1-2):433–450, 2001.

A. Cichon.
Termination orderings and complexity characterizations.
In Proof Theory. Cambridge University Press, 1993.

Theorem

The derivational complexity induced by MPO and LPO is bounded by the slow-growing hierarchy indexed by α.

Proof.

Hofbauer and Weiermann's results, in conjunction with the Hierarchy comparison theorem

$$\mathrm{PRIMREC} = \bigcup_{\alpha < \varphi(\omega,0)} E(\mathrm{G}_\alpha) \qquad \mathrm{MREC} = \bigcup_{\alpha < \Lambda} E(\mathrm{G}_\alpha)$$

NB: Cichon's proof is (unrepairable) wrong ∎

Buchholz's Alternative Proof

The case of LPO

let $\mathcal{F}$ be finite and $>$ a precedence on $\mathcal{F}$

Definition

by $\mathcal{A}(\succ, s, t)$ we denote the following proposition: $s = f(s_1, \ldots, s_n)$ and either

1. $\exists\, i\ s_i \succ t$ or $s_i = t$,
2. $t = g(t_1, \ldots, t_m)$ and $f > g$ and $\forall\, j\ s \succ t_j$, or
3. $t = f(t_1, \ldots, t_n)$ and $\exists\, i$ (i) $\forall\, j \in [1, i-1]$, $s_j = t_j$, (ii) $s_i \succ t_i$ and (iii) $\forall\, j > i\ s \succ t_j$

Lemma

the lexicographic path order $>_{\mathrm{lpo}}$ is the least binary relation $\succ$, st. for all $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $\mathcal{A}(\succ, s, t) \rightarrow s \succ t$

Well-foundedness of LPO

we write $\succ$ as abbreviation for $>_{\text{lpo}}$; let W denote the accessible part of $(\mathcal{T}(\mathcal{F},\mathcal{V}),\prec)$

$$W := \bigcap \{X \subseteq \mathcal{T}(\mathcal{F},\mathcal{V}) \mid \forall t\, (\forall s \prec t\, (s \in X) \to t \in X)\}$$

Lemma

(W1) $\forall t\, (\forall s \prec t\, (s \in W) \leftrightarrow t \in W)$

(W2) $\forall t \in W\, (\forall s \prec t\, F(s) \to F(t)) \to \forall t \in W\, F(t)$, *for each (predicate) formula F*

Lemma

$$\forall t_1,\ldots,t_n \in W\ \left(\forall s_1,\ldots,s_n \in W\, ((s_1,\ldots,s_n) \prec^{lex} (t_1,\ldots,t_n) \to G(s_1,\ldots,s_n)) \to \right.$$
$$\left. \to G(t_1,\ldots,t_n)\right)$$

Lemma

for all $t_1, \dots, t_n \in W$, $g \in \mathcal{F}$, *we have* $g(t_1, \dots, t_n) \in W$

Proof.

by induction on the (finite) precedence $>$, using the previous lemmas

- let $t_1, \dots, t_n \in W$ and
  $\forall s_1, \dots, s_n \in W \; (s_1, \dots, s_n) \prec^{lex} (t_1, \dots, t_n) \rightarrow g(s_1, \dots, s_n) \in W$
- by side-induction on s, we prove $s \prec g(t_1, \dots, t_n)$ implies $s \in W$
  1. suppose $s \preceq t_j$; then $s \in W$ as $t_j \in W$ by **(W1)**
  2. suppose $s = f(s_1, \dots, s_m))$, $f < g$ and for all i, $s_i \prec g(t_1, \dots, t_n)$; then by SIH $s_i \in W$ and by MIH $s \in W$
  3. finally, if $s = g(s_1, \dots, s_n)$ then $(s_1, \dots, s_n) \prec^{lex} (t_1, \dots, t_n)$ and by SIH we have $s_i \in W$; thus by the assumption above $s \in W$
- in sum by **(W1)**, we have $g(t_1, \dots, t_n) \in W$

∎

Lemma

for all $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $t \in W$

Proof.

by induction on the (finite) precedence $>$ ∎

Corollary

there is no infinite $\prec$-decending sequence

Proof.

- by **(W2)** we conclude for each $t \in W$: there exists no infinite $\prec$-descending sequence
- due to the lemma, for all $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $t \in W$

∎

Proof Theoretic Analysis

*If $\prec$ is a primitive recursive relation on $\mathcal{T}(\mathcal{F},\mathcal{V})$ such that Π^0_2-IA proves $\forall s, t\,(s \prec t \to \mathcal{A}(\succ, s, t))$ and if W is a Σ^0_1-set st. Π^0_2-IA proves **(W1)** and **(W2)** for all Π^0_2-formulas $F(t)$, then the well-foundness proof above can be formalised in Π^0_2-IA, and thus Π^0_2-IA proves $\forall t\,(t \in W)$.*

let $\mathrm{depth}(t)$ denote the depth of term t

Definition

we define approximations of $>_{\mathrm{lpo}}$; that is, $s \prec_k t$ holds, if

- $\mathcal{A}(\prec_k, s, t)$ and
- $\mathrm{depth}(s) \leqslant k + \mathrm{depth}(t)$

using a standard Gödelisation of terms, we see that $\forall s \prec_k t$ is a bounded quantifier

k-derivations

Definition

$$t \in (t_0, \ldots, t_{n-1}) :\Longleftrightarrow \exists i < n \, (t = t_i)$$
$$\mathcal{D}_k := \{(t_0, \ldots, t_l) \mid \forall j \leqslant l \, \forall s \prec_k t_j \ (s \in (t_0, \ldots, t_l))\}$$
$$W_k := \{t \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid \exists d \, (d \in \mathcal{D}_k \wedge t \in d)\}$$

the elements of $\mathcal{D}_k$ are call k-derivations and note that the W_k are Σ^0_1-sets

Lemma

Π^0_2-IA proves the following

(W_k1) $\forall t \, (\forall s \prec_k t \, (s \in W_k) \leftrightarrow t \in W_k)$

(W_k2) $\forall t \in W_k \, (\forall s \prec_k t \, F(s) \to F(t)) \to \forall t \in W_k \, F(t)$, *for each Π^0_2 formula F*

Minaturisation of Well-foundness Proof

Theorem

- Π^0_2-*IA proves*
$$\forall t \in \mathcal{T}(\mathcal{F},\mathcal{V})\ (t \in W_k) \quad \textit{that is} \quad \forall t \in \mathcal{T}(\mathcal{F},\mathcal{V})\ \exists d (d \in \mathcal{D}_k \land t \in d)$$
- *the length of any* $\prec_k$*-decending chain is bounded by a multiple-recursive function*

Proof.

- for the first claim, we note that the W_k are Σ^0_1-sets and follow the recipe of the (original) well-foundedness proof
- for the second, observe that the provable recursive functions of Π^0_2-IA are the multiple-recursive functions
- the latter is orginally due to Parsons, a modern treatment can be found in Fairtlough and Wainer's handbook article ∎

Derivational Complexity Induced

Lemma

- *if $s >_{\mathsf{lpo}} t$, then $s\sigma \succ_{\mathsf{depth}(t)} t\sigma$ for every substitution σ*
- *let $C[\cdot]$ denote a (term) context, then $s \succ_k t$ implies $C[s] \succ_k C[t]$*

Proof.

we only prove the first claim; suppose $s >_{\mathsf{lpo}} t$; then by induction on $>_{\mathsf{lpo}}$ on first proves that

$$\mathsf{depth}(s\sigma) + \mathsf{depth}(t) \geqslant \mathsf{depth}(t\sigma)$$

and second that $\mathcal{A}(\prec_{\mathsf{depth}(t)}, s, t)$ ∎

let $\mathcal{R}$ be finite TRS such that $\mathcal{R} \subseteq >_{\mathsf{lpo}}$ and let $k := \max\{\mathsf{depth}(r) \mid l \to r \in \mathcal{R}\}$

Corollary

if $s \to_{\mathcal{R}} t$ then $s \succ_k t$ ∎

Buchholz' proof

let $\mathcal{R}$ be a finite TRS and $\mathcal{R} \subseteq >_{\text{lpo}}$

$$t_1 \succ_k t_2 \succ_k t_3 \succ_k \cdots \succ_k t_n$$

∎

what happens, if we directly employ interpretations into ordinals and collapse with the slow-growing hierarchy

Cichon's claim

$$\mathrm{G}_{\mathrm{otyp}(t_1)}(?) > \mathrm{G}_{\mathrm{otyp}(t_2)}(?) > \mathrm{G}_{\mathrm{otyp}(t_3)}(?) > \cdots > \mathrm{G}_{\mathrm{otyp}(t_n)}(?)$$

Remarks

- the collapse only works as intended, if we decend along the a fundamental sequence
- the "slow-growing" hierarchy may be fast-growing!

What Makes a Pointwise Hierarchy Slow Growing?

let's refer to $(G_\alpha)_{\alpha \in \mathcal{O}}$ as the pointwise hierarchy

Theorem

the pointwise hierarchy is slow growing, when the (underlying) fundamental sequence is defined as

$$\lambda[x] = \begin{cases} \omega^\alpha + \lambda'[x] & \text{if } \lambda = \omega^\alpha + \lambda' \\ \omega^\beta \cdot 2^x & \text{if } \lambda = \omega^{\beta+1} \\ \omega^{\lambda'[2^x]} & \text{if } \lambda = \omega^{\lambda'} \end{cases}$$

where $\omega^\alpha + \lambda' > \lambda'$, λ' a limit ordinal

A. Weiermann.
What makes a (pointwise) hierarchy slow growing?
In Sets and Proofs, Invited Papers from the Logic Colloquium 97, page 403–423. 1999.

Theorem

the pointwise hierarchy is fast growing, when the (underlying) fundamental sequence is defined as

$$\lambda[x] = \begin{cases} \omega^\alpha + \lambda'[x+1] & \text{if } \lambda = \omega^\alpha + \lambda' \\ \omega^\beta \cdot (x+1) & \text{if } \lambda = \omega^{\beta+1} \\ \omega^{\lambda'[x]} & \text{if } \lambda = \omega^{\lambda'} \end{cases}$$

where $\omega^\alpha + \lambda' > \lambda'$, λ' a limit ordinal

A. Weiermann.
Sometimes slow growing is fast growing.
Ann. Pure Appl. Log., 90(1-3):91–99, 1997.

T. Arai.
Variations on a theme by Weiermann.
J. Symb. Log., 63(3):897–925, 1998.

Hydra Battle in Rewriting

Example (Dershowitz and Jouannaud)

$$\mathsf{h}(\mathsf{e}(x), y) \to \mathsf{h}(\mathsf{d}(x, y), \mathsf{s}(y)) \qquad (\alpha, n) \Longrightarrow (\alpha_n, n+1)$$

$$\mathsf{d}(\mathsf{g}(\mathsf{g}(0, x), y), 0) \to \mathsf{e}(y) \qquad \text{standard Hydra battle}$$

$$\mathsf{d}(\mathsf{g}(0, x), y) \to \mathsf{e}(x)$$

$$\mathsf{d}(\mathsf{g}(x, y), z) \to \mathsf{g}(\mathsf{d}(x, z), \mathsf{e}(y))$$

$$\mathsf{d}(\mathsf{g}(\mathsf{g}(0, x), y), \mathsf{s}(z)) \to \mathsf{g}(\mathsf{e}(x), \mathsf{d}(\mathsf{g}(\mathsf{g}(0, x), y), z))$$

$$\mathsf{g}(\mathsf{e}(x), \mathsf{e}(y)) \to \mathsf{e}(\mathsf{g}(x, y)) \qquad \text{auxiliary rule}$$

Remark

the given system constitutes the corrected version by Dershowitz; the orginal system by Dershowitz and Jouannaud does not reflect the standard Hydra battle

RTALooP # 23

Must any termination ordering used for proving termination of the Battle of Hydra and Hercules-system have the Howard[-Bachmann] ordinal as its order type?[1]

NB. The conjecture follow's from Cichon's conjecture in conjunction with the Hierarchy comparision theorem

Answer

- No; more precisely, it is relative straightforward to design a reduction order of order type ϵ_0 that proves termination of Dershowitz's corrected system
- a more involved construction is necessary to handle the original system

GM.
The Hydra battle and Cichon's principle.
AAECC, 20(2):133–158, 2009.

[1] `http://www.win.tue.nl/rtaloop/`.

Wrap Up

1. as mentioned, Cichon's conjecture links
   - logical complexities of a termination proof and
   - computational complexities of a given program
2. connection is (a lot more) subtle than envisioned; developed evidence suggest that "slow-growing" is a misnomer, "pointwise" is more apt
3. technically, applicability hinges on
   - the definition of the (underlying) fundamental sequences
   - mapping the given reduction order $\succ$ to a descend along these fundamental sequences

NB. in practise, methodologies for cost analysis of programs are only loosely based on termination techniques; eg. polynomial (or even sublinear) bounds resource bounds forbid arguing "in the large"

universität innsbruck

Further Reading

M. Avanzini.
Verifying Polytime Computability Automatically.
PhD thesis, University of Innsbruck, 2013.

T. Arai.
Consistency proof via pointwise induction.
Arch. Math. Log., 37(3):149–165, 1998.

T. Arai.
Variations on a theme by Weiermann.
J. Symb. Log., 63(3):897–925, 1998.

T. Arai, S. S. Wainer, and A. Weiermann.
Goodstein sequences based on a Parametrized Ackermann-Péter function.
Bull. Symb. Log., 27(2):168–186, 2021.

A. Beckmann and A. Weiermann.
A term rewriting characterization of the polytime functions and related complexity classes.
Arch. Math. Log., 36(1):11–30, 1996.

E.-A. Cichon and A. Weiermann.
Term rewriting theory for the primitive recursive functions.
APAL, 83(3):199–223, 1997.

GM.
The Hydra battle and Cichon's principle.
AAECC, 20(2):133–158, 2009.

GM and A. Weiermann.
Relating derivation lengths with the slow-growing hierarchy directly.
In Proc. 14th RTA, volume 2706, pages 296–310, 2003.

M. Rathjen and A. Weiermann.
Proof-theoretic investigations on kruskal's theorem.
Ann. Pure Appl. Log., 60(1):49–88, 1993.

W. Sieg.
Fragments of arithmetic.
APAL, 28:33–71, 1985.

A. Weiermann.
Complexity bounds for some finite forms of kruskal's theorem.
J. Symb. Comput., 18(5):463–488, 1994.

A. Weiermann.
Sometimes slow growing is fast growing.
Ann. Pure Appl. Log., 90(1-3):91–99, 1997.

A. Weiermann.
What makes a (pointwise) hierarchy slow growing?
In Sets and Proofs, Invited Papers from the Lo- gic Colloquium 97, page 403–423. 1999.

A. Weiermann.
Some Interesting Connections Between The Slow Growing Hierarchy and The Ackermann Function.
J. Symb. Log., 66(2):609–628, 2001.

A. Weiermann.
Slow versus fast growing.
Synth., 133(1-2):13–29, 2002.

universität innsbruck

Thank You for Your Attention!