Mechanizing Meta-Theory in Beluga

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- Proofs (that a given property is satisfied) are becoming pervasive and an integral part of certified software. (see: CompCert, DeepSpec, RustBelt, Sel4, Cogent)

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Meta-Theory

Program
(in Assembler, C, ML, Java, Rust, ...)

Properties:

– **Memory/Type Safety**: Prog. doesn't crash

– **Contextual Equivalence**: Two programs are indistinguishable in any valid program context

– **Bisimulation**: Two systems behave the same

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  - D. Hirschkoff [TPHOLs'97]: Bisimulation Proofs for the π-calculus in Coq (600 out of 800 lemmas are infrastructural)
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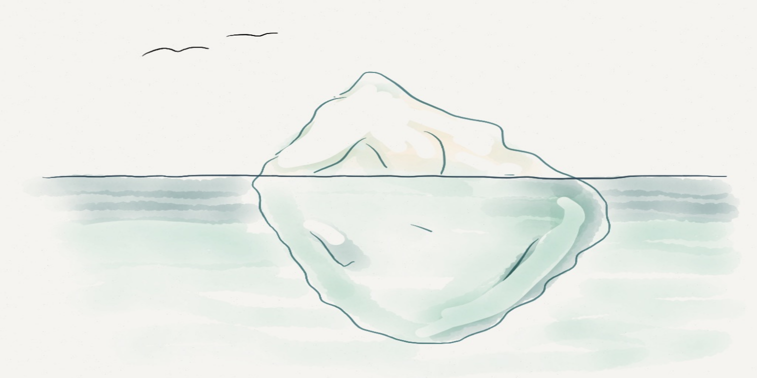
- Scalability, reusability, maintainability, automation

Proofs: The tip of the iceberg



"We may think of [the] proof as an iceberg. In the top of it, we find what we usually consider the real proof; underwater, the most of the matter, consisting of all mathematical preliminaries a reader must know in order to understand what is going on." *S. Berardi [1990]*

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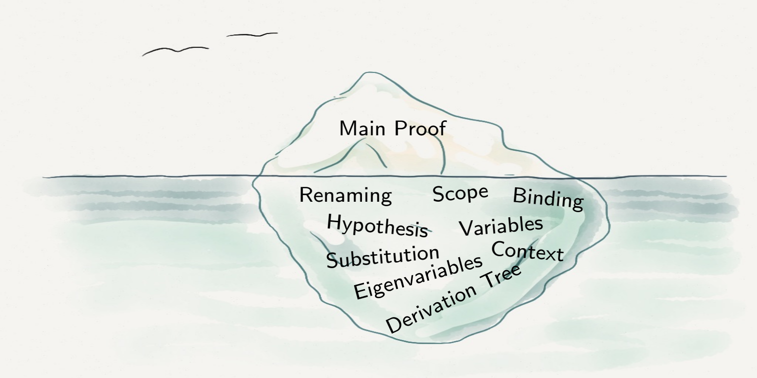

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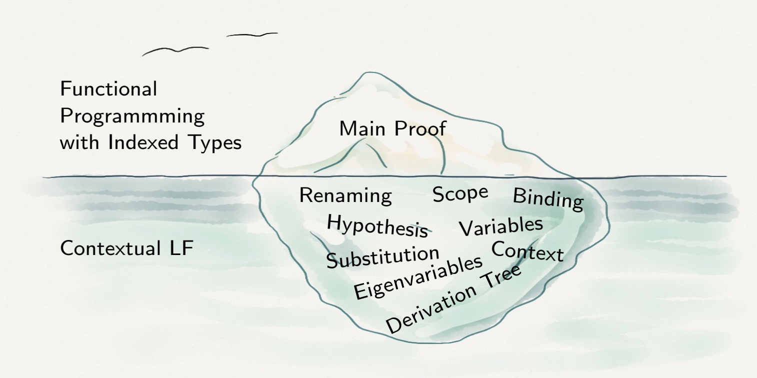
Question

What are good meta-languages to program and reason with formal systems and proofs?

"The motivation behind the work in very-high-level languages is to ease the programming task by providing the programmer with a language containing primitives or abstractions suitable to his problem area. The programmer is then able to spend his effort in the right place; he concentrates on solving his problem, and the resulting program will be more reliable as a result. Clearly, this is a worthwhile goal." *B. Liskov [1974]*

Above and Below the Surface

BELUGA: Dependently typed Programming and Proof Environment



- Below the surface: Support for key concepts based on Contextual LF
- Above the surface: (Co)Inductive Proofs = (Co)Recursive Programs using (Co)pattern Matching with built-in index language of Contextual LF objects

Design of Beluga

- Top : Functional programming with indexed (co)data types [POPL'08,POPL'12,POPL'13,ICFP'16]

On paper proof	In Beluga [IJCAR'10,CADE'15]
Case analysis of inputs	Case analysis via pattern matching
Inversion	Pattern matching using let-expression
Observations on output	Case analysis via copattern matching
(Co)Induction hypothesis	(Co)Recursive call

- Bottom: Contextual LF

Well-formed derivations	Dependent types
Renaming,Substitution	α-renaming, β-reduction in LF
Well-scoped derivation	Contextual types and objects [TOCL'08]
Context	Context schemas
Properties of contexts (weakening, uniqueness)	Typing for schemas
Simultaneous Substitutions (composition, identity)	Substitution type [LFMTP'13,15]

This Talk

Design and implementation of Beluga

- Introduction
- Example: Proof by logical relation
- Writing a proof in Beluga ...
- Conclusion and curent work

"The limits of my language mean the limits of my world."
- L. Wittgenstein

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Simply Typed Lambda-calculus (Gentzen-style)

Types $A, B ::= \mathsf{i} \mid A \Rightarrow B$

Terms M, N $::= x \mid \mathbf{c} \mid \mathsf{lam}\ x.M \mid \mathsf{app}\ M\ N$

Evaluation Judgment: $\boxed{M \longrightarrow M'}$ read as "M steps to M'"

$$\frac{}{\mathsf{app}\ (\mathsf{lam}\ x.M)\ N \longrightarrow [N/x]M}\ \mathsf{s/beta} \qquad \frac{}{M \longrightarrow M}\ \mathsf{s/refl}$$

$$\frac{M \longrightarrow M'}{\mathsf{app}\ M\ N \longrightarrow \mathsf{app}\ M'\ N}\ \mathsf{s/app} \qquad \frac{M \longrightarrow M' \quad M' \longrightarrow N}{M \longrightarrow N}\ \mathsf{s/trans}$$

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Typing Judgment: $\boxed{M : A}$ read as "M has type A" (Gentzen-style)

$$\frac{}{\mathbf{c} : \mathbf{i}}\ \mathrm{const} \qquad \frac{\begin{array}{c}\overline{x : A}\ u \\ \vdots \\ M : B\end{array}}{\mathrm{lam}\, x.M : A \Rightarrow B}\ \mathrm{lam}^{x,u} \qquad \frac{M : A \Rightarrow B \quad N : A}{\mathrm{app}\ M\ N : B}\ \mathrm{app}$$

Simply Typed Lambda-calculus with Contexts

Types and Terms

Types $A, B ::= \mathsf{i} \mid A \Rightarrow B$

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Typing Judgment: $\boxed{\Gamma \vdash M : A}$ read as "M has type A in context Γ"

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \mathsf{lam}\, x.M : A \Rightarrow B}\ \mathsf{lam}^x \qquad \frac{\Gamma \vdash M : A \Rightarrow B \quad \Gamma \vdash N\ A}{\Gamma \vdash \mathsf{app}\ M\ N : B}\ \mathsf{app}$$

Context $\Gamma ::= \cdot \mid \Gamma, x : A$ We are introducing the variable x together with the assumption $x : A$

Weak Normalization for Simply Typed Lambda-calculus

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Proof.

1 Define reducibility candidate $\mathcal{R}_A$

$$\begin{array}{lcl} \mathcal{R}_{\mathbf{i}} & = & \{M \mid M \text{ halts}\} \\ \mathcal{R}_{A \Rightarrow B} & = & \{M \mid M \text{ halts and } \forall N \in \mathcal{R}_A, (\text{app } M\ N) \in \mathcal{R}_B\} \end{array}$$

2 If $M \in \mathcal{R}_A$ then M halts.

3 Backwards closed: If $M' \in \mathcal{R}_A$ and $M \longrightarrow M'$ then $M \in \mathcal{R}_A$.

4 Fundamental Lemma: If $\vdash M : A$ then $M \in \mathcal{R}_A$. (Requires a generalization)

□

Generalization of Fundamental Lemma

Lemma (Main lemma)

If $\mathcal{D} : \Gamma \vdash M : A$ *and* $\sigma \in \mathcal{R}_\Gamma$ *then* $[\sigma]M \in \mathcal{R}_A$.

where $\sigma \in \mathcal{R}_\Gamma$ is defined as:

$$\frac{}{\cdot \in \mathcal{R}_{.}} \qquad\qquad \frac{\sigma \in \mathcal{R}_\Gamma \quad N \in \mathcal{R}_A}{(\sigma, N/x) \in \mathcal{R}_{\Gamma, x:A}}$$

Generalization of Fundamental Lemma

Lemma (Main lemma)

If $\mathcal{D} : \Gamma \vdash M : A$ *and* $\sigma \in \mathcal{R}_\Gamma$ *then* $[\sigma]M \in \mathcal{R}_A$.

Proof.

Case $\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, x{:}A \vdash M : B\end{array}}{\Gamma \vdash \mathrm{lam}\ x.M : A \Rightarrow B}\ \mathit{lam}$

$[\sigma](\mathrm{lam}\ x.M) = \mathrm{lam}\ x.([\sigma, x/x]M)$ by properties of substitution

`halts` $(\mathrm{lam}\ x.[\sigma, x/x]M)$ since it is a value

Suppose $N \in \mathcal{R}_A$.

$[\sigma, N/x]M \in \mathcal{R}_B$ by I.H. on $\mathcal{D}_1$ since $\sigma \in \mathcal{R}_\Gamma$

$[N/x][\sigma, x/x]M \in \mathcal{R}_B$ by properties of substitution

$\mathrm{app}\ (\mathrm{lam}\ x.\ [\sigma, x/x]M)\ N \in \mathcal{R}_B$ by Backwards closure

Hence $[\sigma](\mathrm{lam}\ x.M) \in \mathcal{R}_{A \Rightarrow B}$ by definition

Challenging Benchmark

"I discovered that the core part of the proof (here proving lemmas about CR) is fairly straightforward and only requires a good understanding of the paper version. However, in completing the proof I observed that in certain places I had to invest much more work than expected, e.g. proving lemmas about substitution and weakening." *T. Altenkirch [TLCA'93]*

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Step 1: Represent Types and Lambda-terms in LF

Types $A, B ::= \mathsf{i} \mid A \Rightarrow B$

Terms M, N $::= x \mid \mathbf{c} \mid \mathsf{lam}\ x.M \mid \mathsf{app}\ M\ N$

Typing rules

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LF representation in Beluga

```
LF tp:type =
| i: tp
| arr: tp → tp  → tp;
```

```
LF tm: tp → type =
| c   : tm i
| lam:(tm A → tm B)  → tm (arr A B)
| app: tm (arr A B) → tm A → tm B;
```

- Higher-order abstract syntax (HOAS) to represent variabe binding (lam x.app (lam $y.y$) x) is represented as `(lam λx. app (lam λy.y) x)`.
- Inheriting α-renaming and single substitutions (β-reduction) from LF

Step 1: Encoding Evaluation in LF

Evaluation Judgment: $\boxed{M \longrightarrow M'}$ read as "M steps to M'"

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LF representation in Beluga

```
LF step: tm A → tm A → type =
| s/beta : step (app (lam M) N) (M N)
| s/app  : step M M' → step (app M N) (app M' N)
| s/refl : step M M
| s/trans: step M M' → step M' N → step M N;
```

- Substitution in the `tm` language is modelled via LF application and LF β-reduction

So far . . .

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Question: How to reason about LF terms and types?

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Question: How to reason about LF terms and types?

Answer: Contextual terms and types [TOCL'08]

What are contextual terms and types?

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- Subexpression `app (lam` λ`y.y) x` refers to the variable `x`!
- **The contextual view**:
  Pair up terms and types with their context of variables!

  `[x:tm _` $\vdash$ `app (lam` λ`y.y) x]` has type `[x:tm _` $\vdash$ `tm _ ]`

- Contextual terms and types are closed objects!
  $\Longrightarrow$ there are canonical forms
  $\Longrightarrow$ check for equality by comparing their canonical forms

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- Contextual terms and types are closed objects!
  $\Longrightarrow$ there are canonical forms
  $\Longrightarrow$ check for equality by comparing their canonical forms
- Reason about contextual terms and types using first-order logic with least and greatest fixed points.
  $\Longrightarrow$ need to abstract over contexts

Reducibility Candidates as Indexed Types

Reducibility candidates for terms $M \in \mathcal{R}_A$:

$$\begin{array}{lcl} \mathcal{R}_{\mathbf{i}} & = & \{M \mid \texttt{halts}\ M\} \\ \mathcal{R}_{A \Rightarrow B} & = & \{M \mid \texttt{halts}\ M \text{ and } \forall N \in \mathcal{R}_A, (\mathsf{app}\ M\ N) \in \mathcal{R}_B\} \end{array}$$

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Computation-level data types in Beluga

```
stratified   Reduce : {A:[tp]} {M:[tm A]} type =
| I   : [halts M] → Reduce [i] [M]
| Arr : [halts M] →
        ({N:[tm A]} Reduce [A] [N] → Reduce [B] [app M N])
     → Reduce [arr A B] [M];
```

- `[app M N]` and `[arr A B]` is shorthand for `[ ⊢app M N]` and `[ ⊢arr A B]`; they are contextual types [TOCL'08].
- Note: → is overloaded.
  - → is the LF function space : binders in the object language are modelled by LF functions (used inside `[ ]`)
  - → is a computation-level function (used outside `[ ]`)
- Not strictly positive definition, but stratified.

Reducibility Candidates as Indexed Types

Reducibility candidates for substitutions $\sigma \in \mathcal{R}_\Gamma$:

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Computation-level data types in Beluga

```
inductive  RedSub : (Γ:ctx){σ: ⊢ Γ} type =
| Nil  : RedSub [ ⊢ ^ ]
| Cons : RedSub [ ⊢ σ] → Reduce [ ⊢ A] [ ⊢ M] → RedSub [ ⊢ σ, M ];
```

- Contexts are structured sequences and are classified by context schemas

  ```
  schema ctx =  x:tm A.
  ```

- Substitution σ are first-class and have type $\Psi \vdash \Phi$ providing a mapping from Φ to Ψ.

Theorems as Computation-level Types

Lemma (Backward closed)

If $M \longrightarrow M'$ *and* $M' \in \mathcal{R}_A$ *then* $M \in \mathcal{R}_A$.

```
rec closed : [step M M’] → Reduce [A] [M’] → Reduce [A] [M] = ? ;
```

Lemma (Main lemma)

If $\Gamma \vdash M : A$ *and* $\sigma \in \mathcal{R}_\Gamma$ *then* $[\sigma]M \in \mathcal{R}_A$.

```
rec main : {Γ:ctx}{M:[Γ ⊢ tm A[]]} RedSub [ ⊢ σ] → Reduce [A] [M[σ]] = ? ;
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Fundamental Lemma

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mlam Γ ⇒ mlam M ⇒ fn rs ⇒ case [Γ ⊢ M ] of
| [Γ ⊢ #p ] ⇒ lookup [Γ] [Γ ⊢ #p ] rs                         % Variable
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| [Γ ⊢ app M1 M2] ⇒                                      % Application
  let Arr ha f = main [Γ] [Γ ⊢ M1] rs in
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| [Γ ⊢ lam λx. M1] ⇒                                       % Abstraction
  Arr [ ⊢ h/value s/refl v/lam]
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```

- Direct encoding of on-paper proof
- Equations about substitution properties automatically discharged (amounts to roughly a dozen lemmas about substitution and weakening)
- Total encoding about 75 lines of Beluga code

Some Alternatives

General Theorem Proving Environments

- Calculus of Construction (Coq) / Martin Löf Type Theory (Agda)
  No special support for variables, assumptions, derivation trees, etc.
  About a dozen extra lemmas
- Isabelle / Nominal
  support for variable names, but not for assumptions, derivation trees, etc.
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  based on nominal set theory; about a dozen extra lemmas

Domain-specific Provers (Higher-Order Abstract Syntax (HOAS))

- Abella: encode second-order hereditary Harrop (HH) logic in $\mathcal{G}$, an extension of first-order logic with a new quantifier ∇, and develop inductive proofs in $\mathcal{G}$ by reasoning about the size of HH derivations .
  diverges a bit from on-paper proof; 4 additional lemmas
- Twelf: Too weak for directly encoding such proofs; implement auxiliary logic.

Logical Relations on Open Terms

- Allowing reductions under lambda-abstractions:

$$\frac{M \longrightarrow N}{\mathsf{lam}\, x.M \longrightarrow \mathsf{lam}\, x.N}\ \mathsf{s/lam}^x$$

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- Revisiting the reducibility candidates
  for well-scoped and well-typed open terms $\Gamma \vdash M \in \mathcal{R}_A$:

$$\begin{array}{lcl}
\mathcal{R}_{\mathbf{i}} & = & \{\Psi \vdash M \mid \Psi \vdash M \text{ halts}\} \\
\mathcal{R}_{A \Rightarrow B} & = & \{\Psi \vdash M \mid \Psi \vdash M \text{ halts and } \forall \Phi \geq_\rho \Psi, N \text{ where } \Phi \vdash N : A, \\
 & & \quad \text{if } (\Phi \vdash N) \in \mathcal{R}_A \text{ then } (\Phi \vdash \mathsf{app}\ M[\rho]\ N) \in \mathcal{R}_B\}
\end{array}$$

Encoding Logical Relations on Open Terms

Definition on paper:

$$\begin{aligned}
\mathcal{R}_{\mathbf{i}} &= \{\Psi \vdash M \mid \Psi \vdash M \text{ halts}\} \\
\mathcal{R}_{A \Rightarrow B} &= \{\Psi \vdash M \mid \Psi \vdash M \text{ halts and } \forall \Phi \geq_\rho \Psi, N \text{ where } \Phi \vdash N : A, \\
&\qquad \text{if } (\Phi \vdash N) \in \mathcal{R}_A \text{ then } (\Phi \vdash \text{app } M[\rho]\ N) \in \mathcal{R}_B\}
\end{aligned}$$

Encoding in Beluga

```
stratified   Reduce : (Ψ:nctx) {A:[tp]} {M:[Ψ ⊢  tm A[]]} type =
| Base : Halts [i] [Ψ  ⊢  M] → Reduce [i] [Ψ ⊢  M]
| Arr :  {M:Ψ  ⊢  tm (arr A[] B[])]}
         Halts [arr A B] [Ψ ⊢ M] →
         ({Φ:nctx} {ρ:[Φ ⊢ Ψ]} {N:[Φ ⊢ tm A[]]}
           Reduce [A] [Φ ⊢  N] → Reduce [B] [Φ ⊢  app M[ρ] N])
         → Reduce [arr A B] [Ψ ⊢  M];
```

See our journal paper discussing case studies [MSCS'16]

POPLMark Reloaded!

Strong normalization of a simply-typed lambda-calculus using Kripke-style logical relations.

POPLMark Reloaded: Goal

Benchmark problems that

- Push the state of the art in the area and outline new areas of research
- Compare systems and mechanized proofs qualitatively
- Understand what infrastructural parts (boilerplate) should be generically supported and factored
- Find bugs in existing proof assistants
- Highlight theoretical limitations of existing proof environments
- Highlight practical limitations of existing proof environments
- Polularize and provide a better understanding of logical relations

Question

Why pick strong normalization for simply-typed lambda-calculus using Kripke-style logical relations?

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Why pick strong normalization for simply-typed lambda-calculus using Kripke-style logical relations?

Follow up:

We can prove SN without Kripke-style logical relations and we've already done it.

Witness 1: Lego [Altenkirch'93]

... "following Girard's Proofs and Types"

Characteristic Features:

- Terms are not well-scoped or well-typed
- Candidate relation is untyped and does not enforce well-scoped terms
  $\Longrightarrow$ does not scale to typed-directed evaluatation or equivalence
  $\Longrightarrow$ maybe better techniquues to modularize and structure proof

Witness 2: Abella, ATS/HOAS

... "following Girard's Proofs and Types"

Witness 2: Abella, ATS/HOAS

... "following Girard's Proofs and Types"

- Strictly speaking:

  SN for simply-typed λ-calculus plus one constant that has any type.

- Adding a constant significantly simplifies the proof
- Reducibility of terms only defined on closed terms
- Strictly speaking:

  Show that SN for simply-typed λ-calculus plus one constant implies also SN for open simply-typed λ-terms

A Call for Action

- Be part of formulating and tackling the challenge
- Choose your favorite proof assistant and complete the challenge

Status Report

- Prototype in OCaml (ongoing - last release March 2015)
  providing an interactive programming mode, totality checker [CADE'15]

  `https://github.com/Beluga-lang/Beluga`

- Mechanizing Types and Programming Languages - A companion:

  `https://github.com/Beluga-lang/Meta`

Current and Future Directions

- Lincx: A linear logical framework LF with first-class contexts (A.L. Georges, A. Murawska, S. Otis)[ESOP'17]
- Programming with syntax in existing proof and programming environments (F. Ferreira[ESOP'17])
  Translate contextual objects via a deep embedding
- Coinductive Proofs (e.g. Contextual Equivalence)[ICFP'16]
  (joint work with A. Momigliano, D. Thibodeau [Dale's Festschrift'17])
- Full Dependent Type Theory with Contextual Objects and First-class Contexts (joint work with A. Abel, F. Ferreira, D. Thibodeau, R. Zucchini)

The End

Thank you!

Download prototype and examples at

`http://complogic.cs.mcgill.ca/beluga/`

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"A language that doesn't affect the way you think about programming, is not worth knowing." - *Alan Perlis*