# Uniform Resource Analysis by Rewriting

## Strengths and Weaknesses

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# Outline

# A Textbook Example

## Example

```
let rec fold_left f acc = function
    [] → acc
  | x::xs → fold_left f (f acc x) xs ;;
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$$\begin{aligned}\mathsf{rev}(1 :: 2 :: 3 :: \mathsf{nil}) &\to_{\mathcal{R}} \mathsf{rev}'(1 :: 2 :: 3 :: \mathsf{nil}, \mathsf{nil})\\ &\to_{\mathcal{R}} \mathsf{rev}'(2 :: 3 :: \mathsf{nil}, 1 :: \mathsf{nil})\\ &\to^*_{\mathcal{R}} \mathsf{rev}'(\mathsf{nil}, 3 :: 2 :: 1 :: \mathsf{nil})\\ &\to_{\mathcal{R}} 3 :: 2 :: 1 :: \mathsf{nil}\end{aligned}$$

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we define the runtime complexity wrt. a terminating $\mathcal{R}$:

$$\mathrm{dh}_{\mathcal{R}}(t) = \max\{n \mid \exists u\ t \to_{\mathcal{R}}^{n} u\}$$

$$\mathrm{rc}_{\mathcal{R}}(n) = \max\{\mathrm{dh}_{\mathcal{R}}(t) \mid \text{size of } t \leqslant n \text{ and } t \text{ is basic}\}$$

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## Example (continued)

we obtain $\mathrm{rc}_{\mathcal{R}} \in \mathrm{O}(n)$

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ahem, actually no

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## Theorem

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D. Hofbauer and C. Lautemann.
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In *Proc. 3rd RTA*, volume 355 of *LNCS*, pages 167–177, 1989.

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Termination proofs by multiset path orderings imply primitive recursive derivation lengths.
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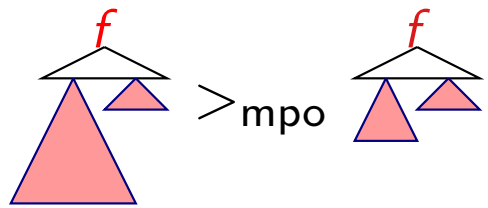

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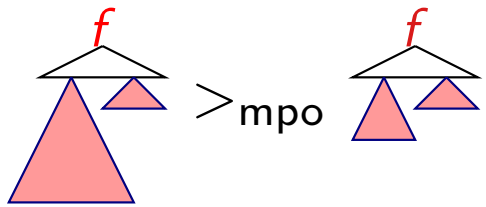
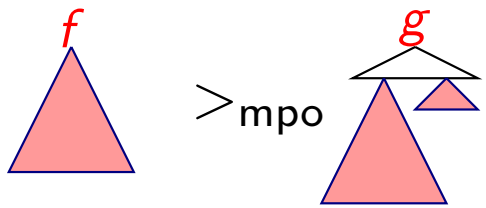



3. $t = g(t_1, \ldots, t_m)$ with $f > g$ and $\forall i\ s >_{\mathsf{mpo}} t_i$



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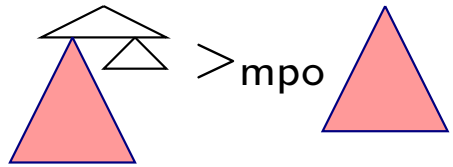
1. $\exists i\ s_i \geqslant_{\mathsf{mpo}} t$

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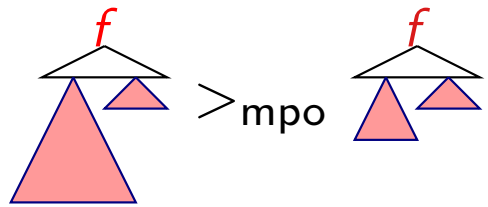


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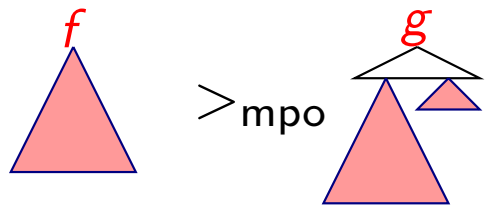

## Theorem

- *polynomial interpretations induce double-exponential derivational complexity*
- *lexicographic path orders induce multiple recursive functions*
- *DP framework in conjunction with the subterm criterion induces multiple recursive functions*
- *simplification orders induce functions elementary in* $\mathsf{H}_\Lambda$*, where* $\Lambda \ggg \epsilon_0$

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I. Lepper.
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A. Weiermann.
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## The Hydra Battle by Kirby and Paris

1. the beast is a finite tree, each leaf corresponds to a head; Hercules chops off heads of the Hydra, but the Hydra regrows:
   - if the cut head has a pre-predecessor, then the remaining subtree issued from this node is multiplied by the stage of the game.
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*termination of the battle is independent: PA $\nvdash$ the battle terminates*

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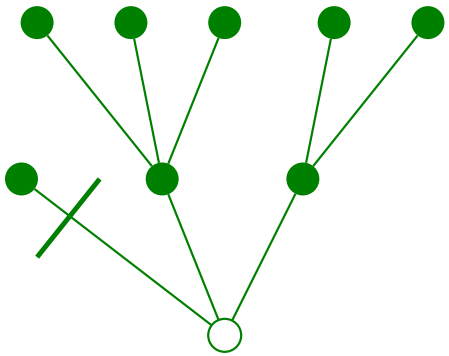
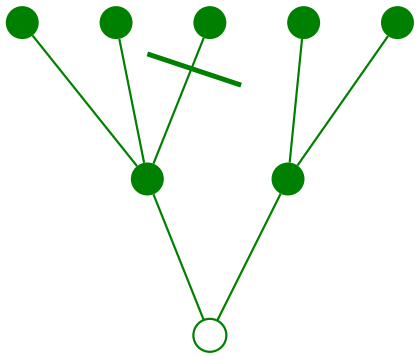
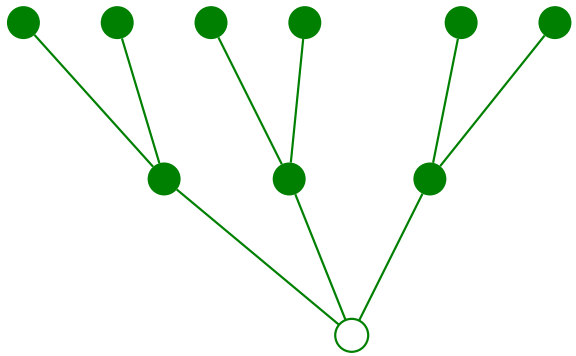


$(H_1, 1)$



$(H_2, 2)$



$(H_3, 3)$

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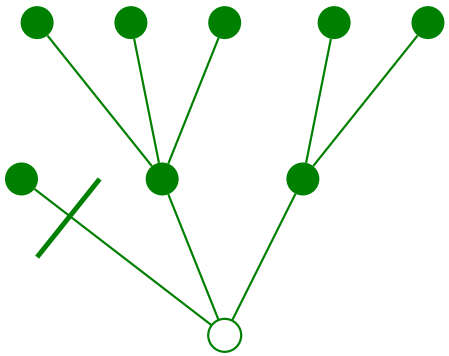
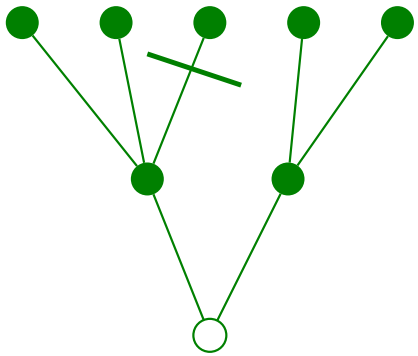


$(\omega^3 \oplus \omega^2 \oplus 1, 1)$



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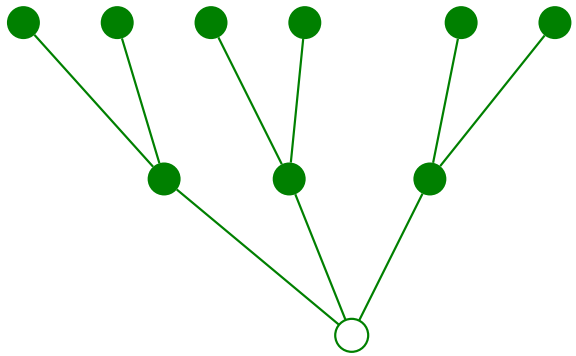

$(\omega^2 \cdot 3, 3)$

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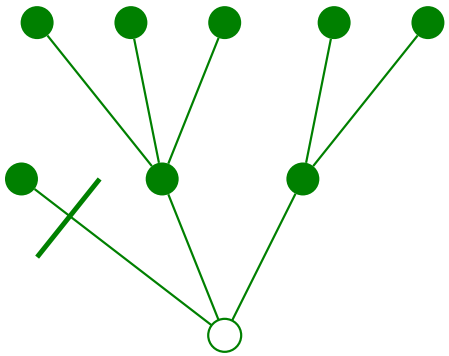
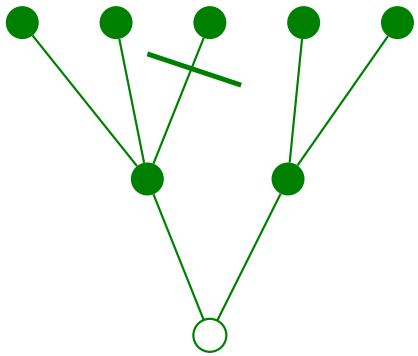


$(\omega^3 \oplus \omega^2 \oplus 1, 1)$

$\omega^3 + \omega^2 + 1$

$>$



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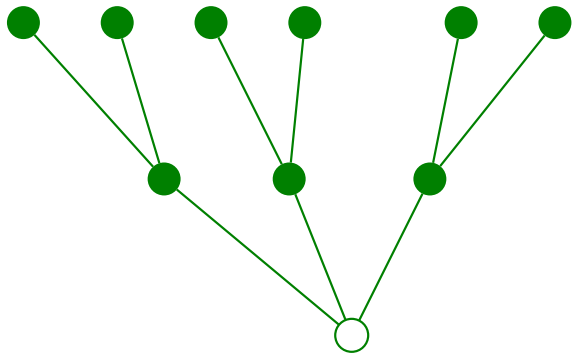

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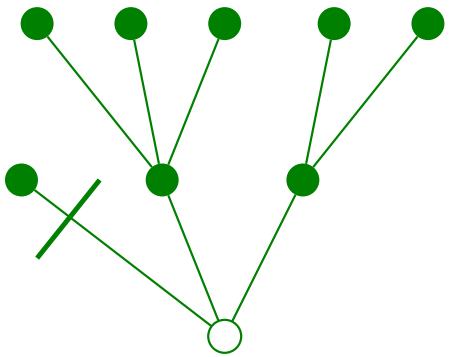

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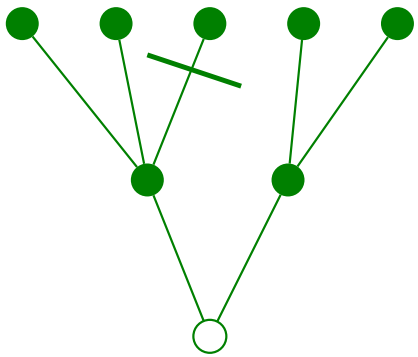
$\succ_{ed}$



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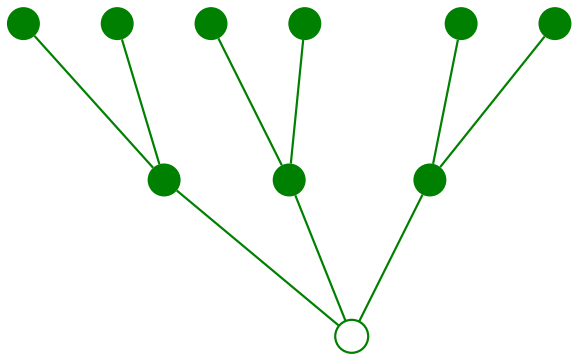

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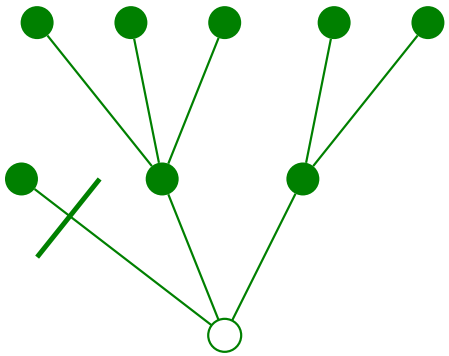

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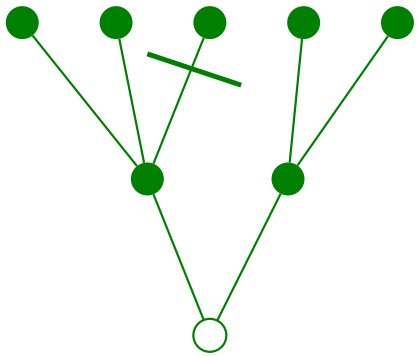

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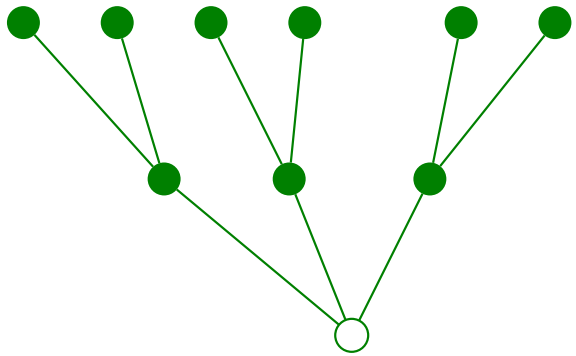

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- based on the game, Dershowitz and Jouannaud designed the TRS as a challenging termination problem
- however, the implementation is buggy and has later been rectified by Dershowitz

## Example

$\mathsf{h}(\mathsf{e}(x), y) \to \mathsf{h}(\mathsf{d}(x, y), \mathsf{s}(y))$ $\qquad (\alpha, n) \Longrightarrow (\alpha_n, n+1)$

$\mathsf{d}(\mathsf{g}(\mathsf{g}(0, x), y), 0) \to \mathsf{e}(y)$ $\qquad$ strategy of the game

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$\mathsf{g}(\mathsf{e}(x), \mathsf{e}(y)) \to \mathsf{e}(\mathsf{g}(x, y))$ $\qquad$ auxiliary rule

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$(\alpha, n) \Longrightarrow (\alpha_n, n + 1)$

strategy of the game

## Remark

- based on the game, Dershowitz and Jouannaud designed the TRS as a challenging termination problem
- however, the implementation is buggy and has later been rectified by Dershowitz

## RTALooP # 23

*Must any termination ordering used for proving termination of the Battle of Hydra and Hercules-system have the Howard[-Bachmann] ordinal as its order type?*[a]

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## Answer

No.

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GM.
The Hydra battle and Cichon's principle.
*AAECC*, 20(2):133–158, 2009.

says

aha, coming back to runtime complexity, I found the following . . .

Tyrolean Complexity Tool - Web Interface - Mozilla Firefox

File Edit View History Bookmarks Tools Help

Tyrolean Complexity Tool

colo6-c703.uibk.ac.at/tct/tct-trs/#output

CBR Home | TcT Home | Download | Experiments | TCT Web

**Input** *(in xml or trs format)*

select example ... | or upload file | Browse... | No file selected.

```
1 (VAR x xs acc)
2
3 (RULES
4   rev(xs) -> revacc(xs,nil)
5   revacc(nil,acc) -> acc
6   revacc(cons(x,xs),acc) -> revacc(xs,cons(x,acc))
7 )
```

**Category**

Complexity Measure: (•) Runtime Complexity ( ) Derivational Complexity

Rewriting Strategy: (•) Full Rewriting ( ) Innermost Rewriting

**Search Strategy**

(•) Automatic ( ) Certify ( ) Customised by user

check with timeout of 30 seconds.

**Output**

```
WORST_CASE(?,O(n^1))
*** 1 Success [(?,O(n^1))]  ***
    Considered Problem:
      Strict DP Rules:
```

 says

indeed, TCT can fully automatically handle the TRS, for example using a method called matrix interpretations

 says

indeed, TCT can fully automatically handle the TRS, for example using a method called matrix interpretations

## Definition

a matrix interpretation of dimension $n$ is a pair $\mathcal{M} = (\mathbb{N}^n, >)$ such that

1. $\forall\ f \in \mathcal{F}$:

$$f_{\mathcal{M}}(\vec{v}_1, \ldots, \vec{v}_k) = M_1\vec{v}_1 + \ldots + M_k\vec{v}_k + \vec{f}$$

2. the $M_i$ are square matrices such that $(M_i)_{1,1} \geqslant 1$
3. the order $>$ is defined as:

$$(x_1, x_2, \ldots, x_n)^\top > (y_1, y_2, \ldots, y_n)^\top$$

if $x_1 > y_1$ and $\forall\ i \geqslant 2$: $x_i \geqslant y_i$

## Example

consider the TRS $(x \circ y) \circ z \to x \circ (y \circ z)$ together with the matrix interpretation $\mathcal{M}$

$$\circ_{\mathcal{M}}(\vec{x}, \vec{y}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \vec{y} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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we obtain $\forall$ assignments to the variables:

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## Observation

$$t = t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} t_4 \to_{\mathcal{R}} \cdots$$

implies

$$[t] > [t_1] > [t_2] > [t_3] > [t_4] > \cdots$$

for compatible matrix interpretations $\mathcal{M}$

## Definition

let $\mathcal{M}$ denote a matrix interpretation and let

$$M = \max\{M_i \mid M_i \text{ a square matrix used in } f_{\mathcal{M}}\}$$

then $M$ is called maximums matrix of $\mathcal{M}$

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## Theorem

*let $\mathcal{M}$ be a matrix interpretation of dimension $d$ compatible with a TRS $\mathcal{R}$ and let $M$ be the maximum matrix of $\mathcal{M}$, if the spectral radius $\rho(M) \leqslant 1$, then $\mathrm{rc}_{\mathcal{R}} \in \mathrm{O}(n^d)$*

## Proof.

by linear algebra, for example Jordan Normal Form Theorem ■

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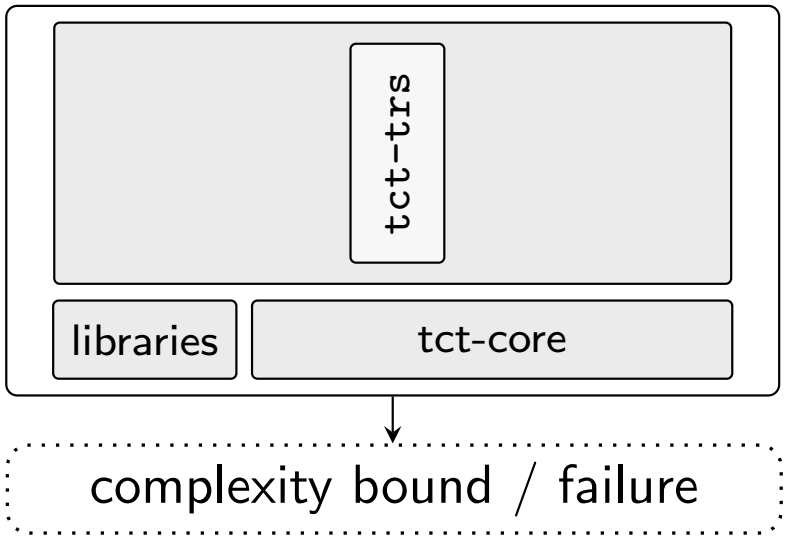
# Tyrolean Complexity Tool (`tct-trs`)

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- modular complexity analysis framework
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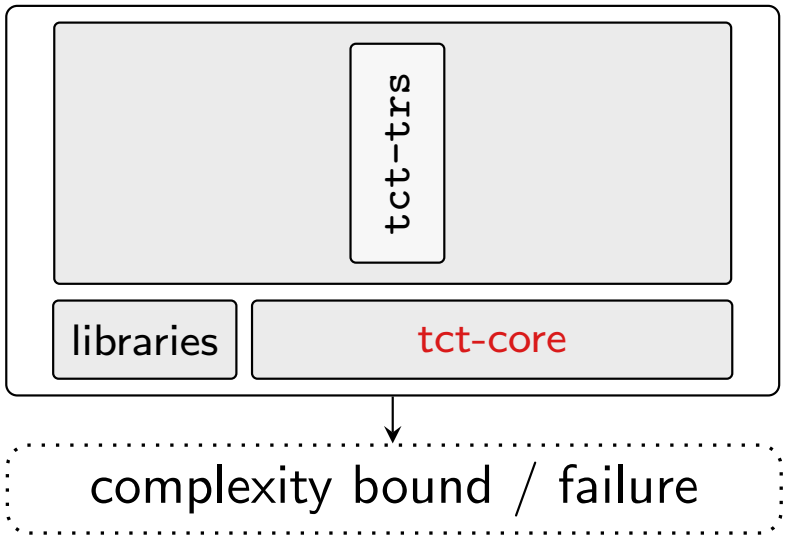


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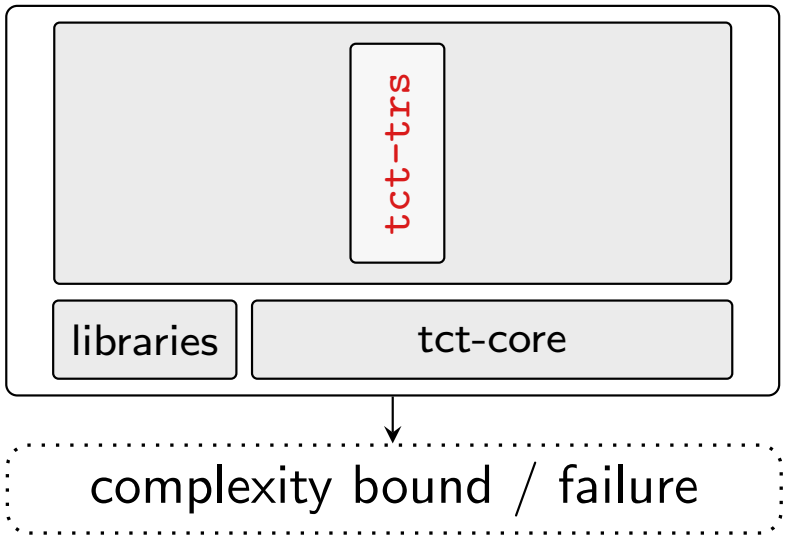


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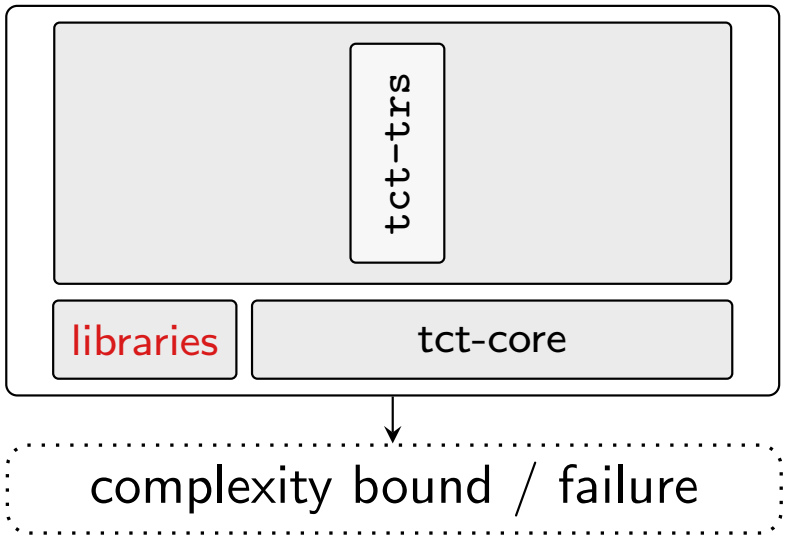


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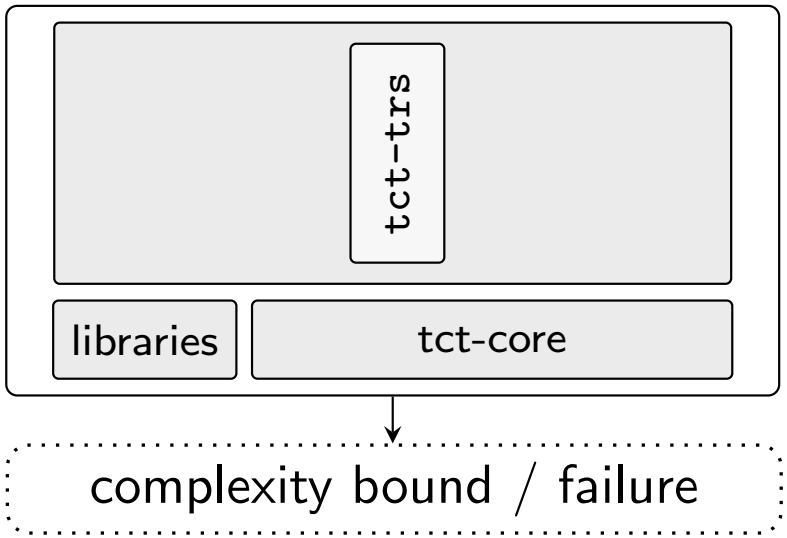


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M. Avanzini, C. Sternagel, and R. Thiemann.
Certification of complexity proofs using CeTA
In *Proc. 26th RTA*, volume 36 of *LIPIcs*, pag…

M. Avanzini and GM.
A combination framework for complexity.
*IC*, 248:22–55, 2016.

M. Avanzini, GM, and M. Schaper.
Tct: Tyrolean complexity tool.
In *Proc. 22nd TACAS*, volume 9636 of *LNCS*, pages 407–423, 2016.


```
let rec fold_left f acc = function
    [] → acc
  | x::xs → fold_left f (f acc x) xs ;;
let rev l = fold_left (fun xs x → x::xs) [] l ;;
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$\mathsf{rev}(xs) \to \mathsf{rev}'(xs, \mathsf{nil})$
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## Definition

- a transformation is complexity reflecting, if upper bound is reflected
- complexity preserving, if lower bound is preserved

# Example Run of tct-hoca

- textbook example `rev`

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- defunctionalised applicative rewrite system

$$
\begin{aligned}
\mathsf{main}(x_0) &\to \mathsf{m}_1(x_0)\ @\ \mathsf{f} & \mathsf{r}(x_0)\ @\ x_1 &\to x_0\ @\ \mathsf{r}_1\ @\ [\,]\ @\ x_1\\
\mathsf{m}_1(x_0)\ @\ x_1 &\to \mathsf{m}_2(x_0)\ @\ \mathsf{r}(x_1) & \mathsf{r}_1\ @\ x_0 &\to \mathsf{r}_2(x_0)\\
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says

the runtime complexity of `rev` is $O(n)$ . . . if TCT is sound

# Experimental Evidence

| | constant | linear | quadratic | polynomial | terminating |
|---|---|---|---|---|---|
| # systems | 2 | 14 | 18 | 20 | 25 |
| HoCA time | 4.56 | 4.56 | 4.56 | 4.56 | 6.48 |
| FOP time | 0.79 | 14.00 | 30.12 | 60.10 | 3.43 |

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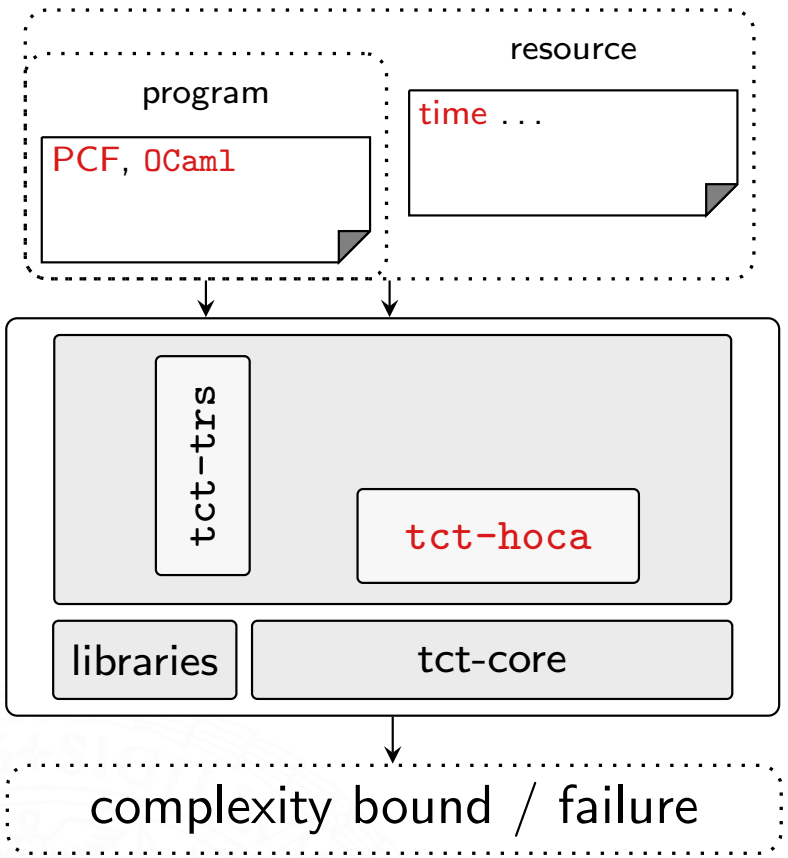
## Testbed

comprises 25 examples, for example including

1. `rev`, `foldl`, `map`, ...
2. merge-sort using a higher-order divide-and-conquer combinator
3. simple parsers relying on the monadic parser-combinator outlined in Okasaki's functional pearl

complexity is tested with TcT; termination with TTT2

# Tyrolean Complexity Tool (`tct-trs`, `tct-hoca`)

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- open source under BSD3
- implemented in Haskell
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- employs SMT-solvers like `minismt` or Z3

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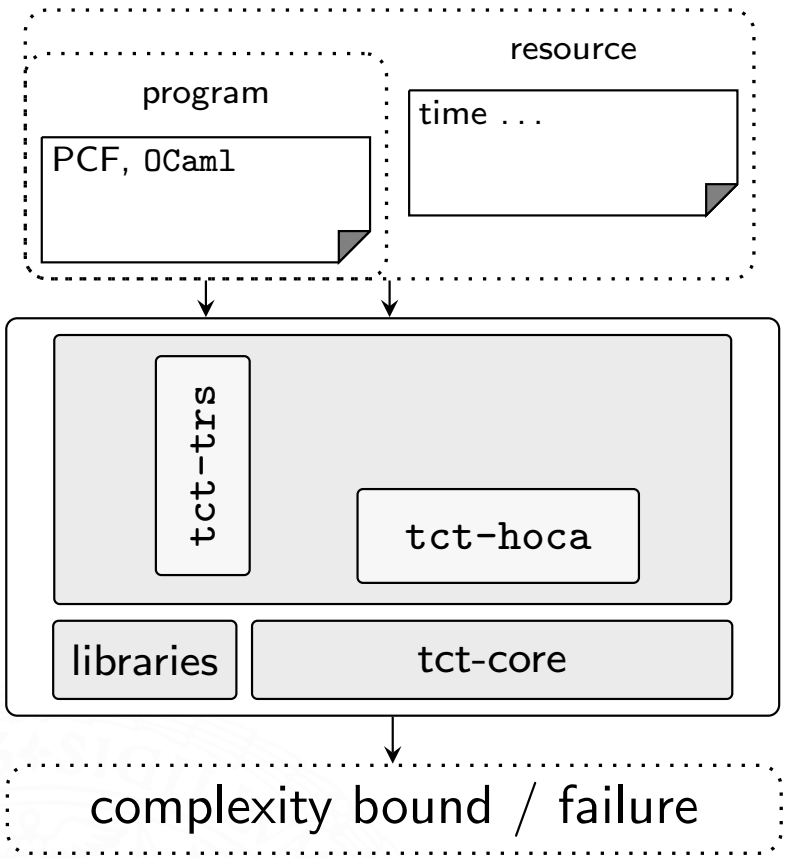


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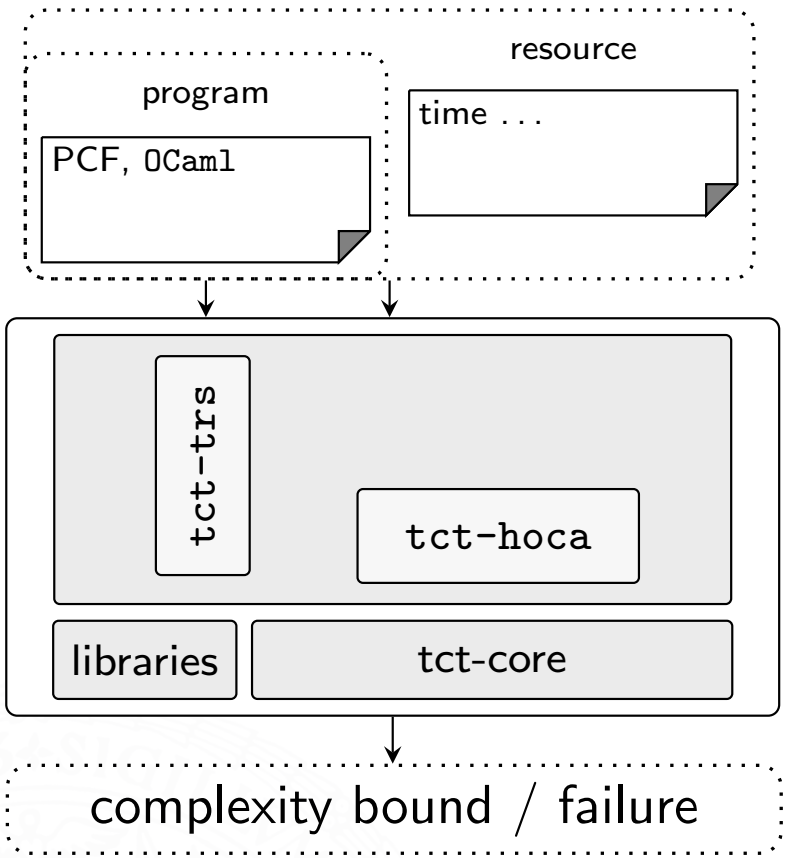


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## Example

```
public static void test(int n, int m){
  if (0 < n && n < m) {
    int j = n+1;
    while(j < n || j > n){
      if (j > m){
        j=0;
      } else {
        j=j+1;
    }}}}
```

## Remarks

- suppose $n < m$: while loop is executed as long as $j \neq n$ holds
- automation requires disjunctive bounds

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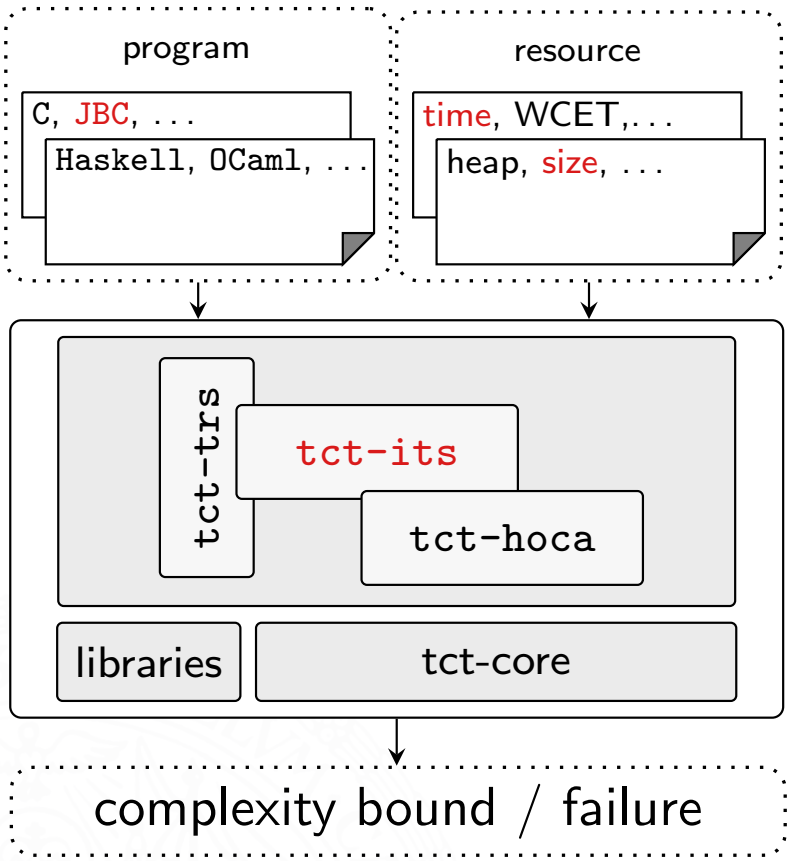
```
start(m,n,j) → while(m,n,n+1) :|: m > n && n > 0
while(m,n,j) → while(m,n,0)   :|: n > 0 && j > n && j > m
while(m,n,j) → while(m,n,j+1) :|: n > 0 && j > n && j ≤ m
while(m,n,j) → while(m,n,j+1) :|: n > 0 && j < n && j ≤ m
```

# Uniform Resource Analysis by Rewriting

## TcT

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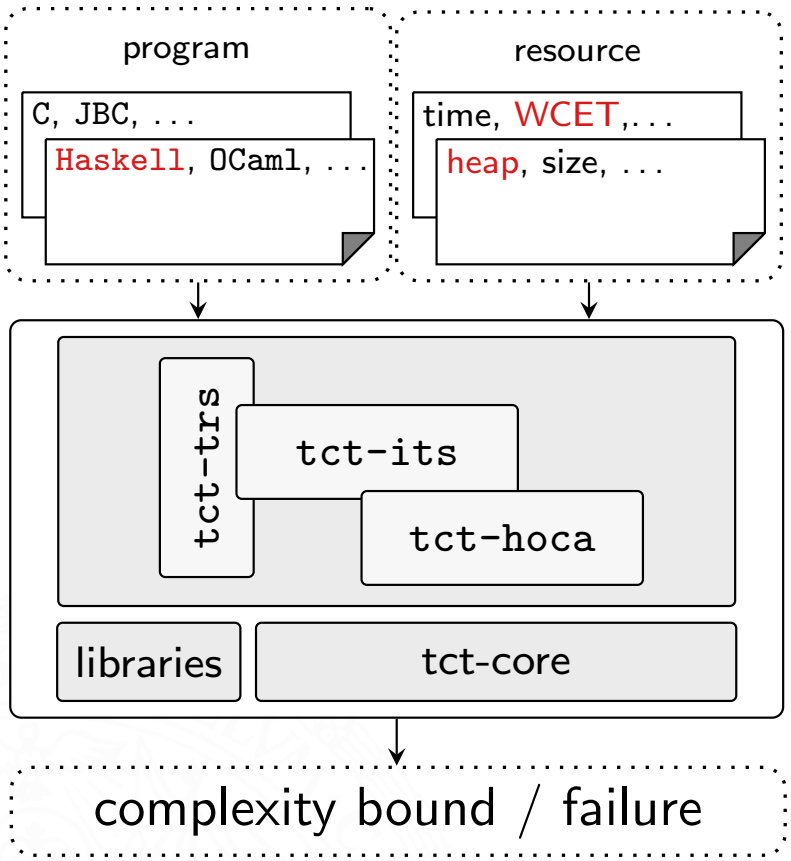


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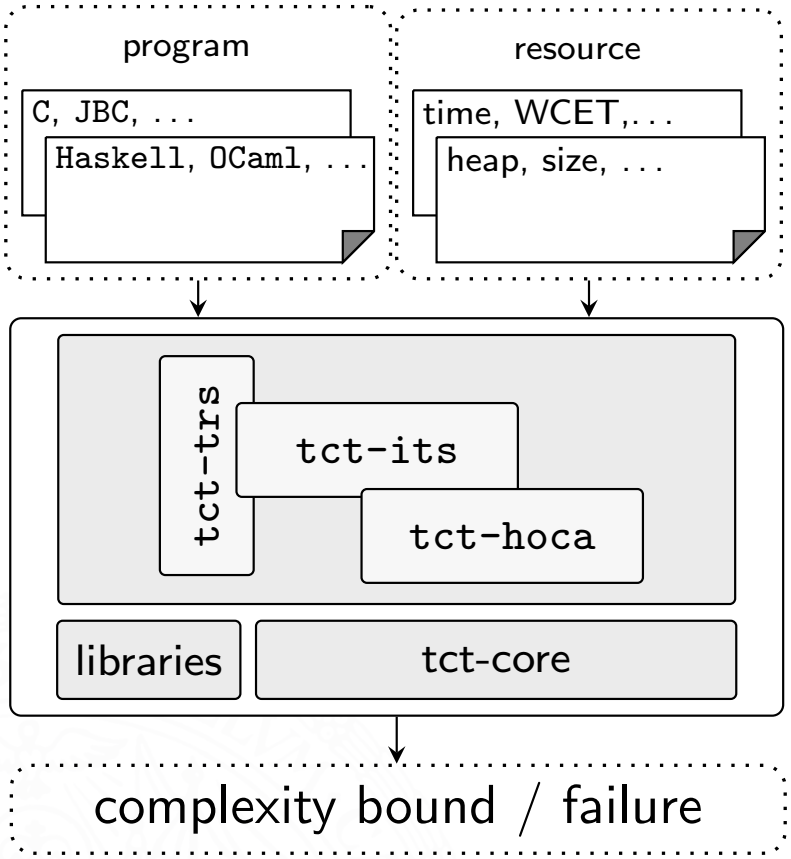
# Uniform Resource Analysis

## TcT, AProVE, CiaoPP, COSTA, SACO, ...

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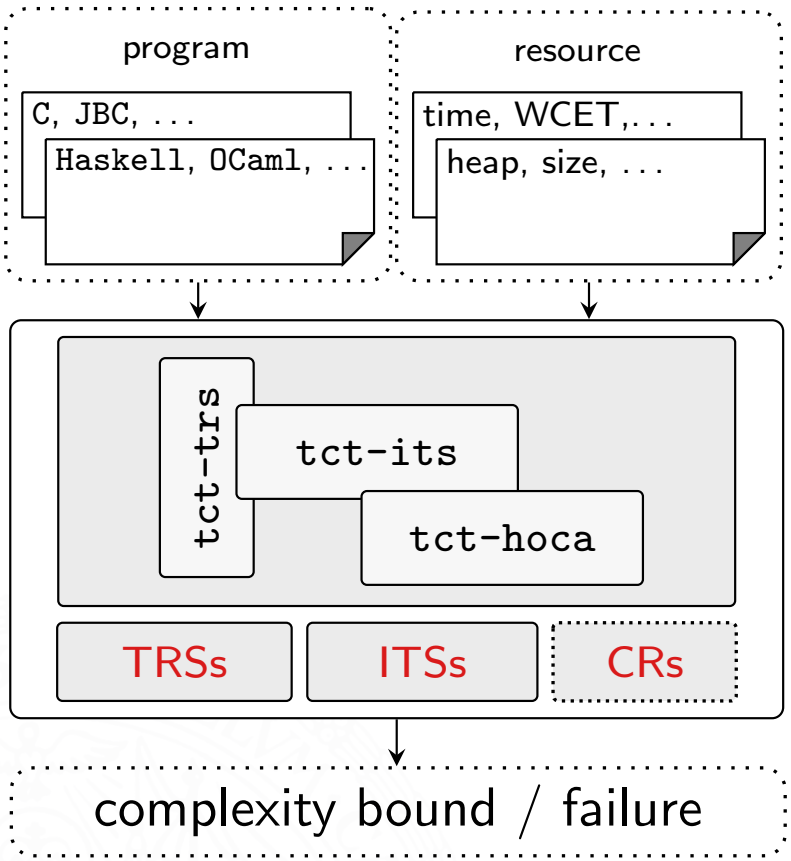
From Jinja Bytecode to Term Rewriting: A Complexity Reflecting Transformation.
*IC*. To appear.

# Uniform Resource Analysis

## TcT, AProVE, CiaoPP, COSTA, SACO, ...




- fully automated resource analysis tool
- competitive for higher-order functional programs, occasionally competitive for object-oriented bytecode programs
- intertwined resource analysis framework

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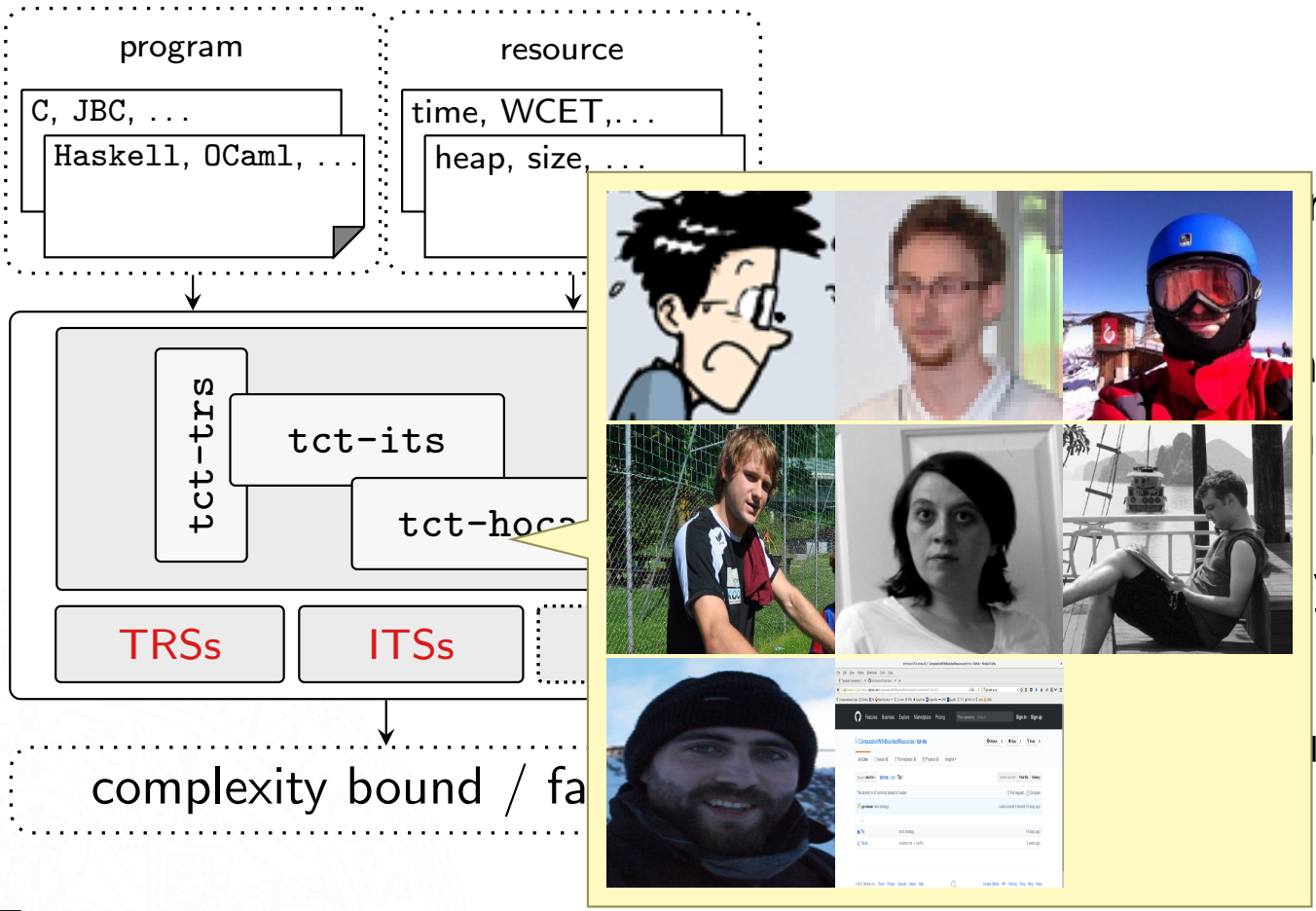


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## Results

- state-of-the-art automated resource analysis tool for higher-order functional programs
- type system based, embodying an amortised analysis
- best-case lower bound and worst-case upper bound analysis
- precise, multivariate bounds

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## Results

- whole program analysis (not composable) for C (LLVM)
- employs vector addition systems with states for program abstraction in conjunction with the synthesis of lexicographic ranking functions
- effective on realistically sized benchmarks and outperforms existing tools (for C or integer transition systems)
- requires the use of additional invariant generations and shape analysis

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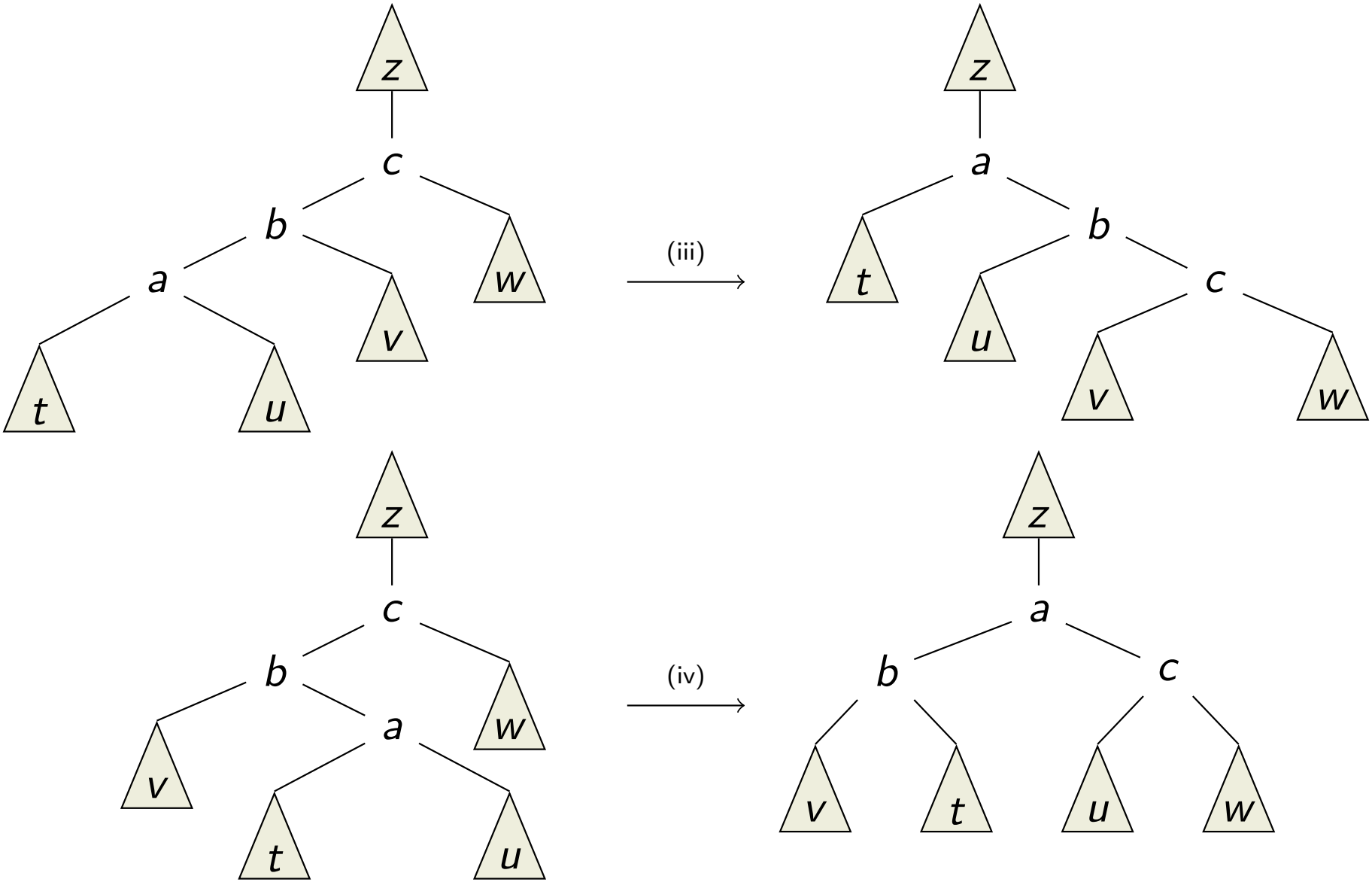
## Challenge by Tobias Nipkow (FSCD'16)

*Martin and Georg: this is all very well what you have done with RaML and TCT, but what about real amortised analyses like Tarjan and Sleator's splay trees?*

# Definition (Splaying, interesting cases)



# Amortised Complexity Analysis

## Definition

the potential of a tree is defined as follows

$$\Phi(\text{nil}) := 0$$
$$\Phi(\langle t, a, u \rangle) := \Phi(t) + \log(|t| + |u|) + \Phi(u)$$

where size $|t|$ denotes the number of leaves of $t$

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$$A(\texttt{splay}\ a\ t) = T(\texttt{splay}\ a\ t) + \Phi(\texttt{splay}\ a\ t) - \Phi(t)$$

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## Lemma (Sleator, Tarjan)

$A(\texttt{splay}\ a\ t) \leqslant 1 + 3\log(|t|)$

# A Type System for Amortised Resource Analysis

## Preprocessing

- assume a size analysis, resulting in a function types over sized types, for example `splay`: $A \times T_n \to T_n$
- the sized type $T_n$ contains trees of exactly size $n$

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## Definitions

- let $\Gamma$ denote a typing context and let $\Gamma \supseteq \{T_{n_1}, \ldots, T_{n_k}\}$
- a size polynomial over $\Gamma$ is a multivariate polynomial over $n_1, \ldots, n_k$
- $p \geqslant q$ iff $\forall n_1, \ldots, n_k$

$$\log(p(n_1, \ldots, n_k)) \geqslant \log(q(n_{i_1}, \ldots, n_{i_l}))$$

- an annotated signature decorates function types with size polynomials

$$\texttt{splay}\colon A \times T_n \xrightarrow{n^3+2} T_n$$

## Definition (Type System (selection))

$$\frac{}{\vdash^{1} \mathsf{nil}: T_1} \qquad \frac{n = k + l}{x_1: T_k, x_2: A, x_3: T_l \vdash^{k \cdot l} \langle x_1, x_2, x_3 \rangle: T_n}$$

$$\frac{\Gamma_1 \vdash^{q_1(\Gamma_1)} e_1: D \quad \Gamma_2, x: D \vdash^{q_2(\Gamma_2)} e_2: C \quad q_1 \cdot q_2 \leqslant p}{\Gamma_1, \Gamma_2 \vdash^{p(\Gamma_1, \Gamma_2)} \texttt{let}\ x = e_1\ \texttt{in}\ e_2: C}$$

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## Definition

P is called well-typed if $\forall\ f(x_1, \ldots, x_k) = e \in \mathrm{P}$ and $\forall\ C_1 \times \cdots \times C_k \xrightarrow{p} C \in \mathcal{F}(f)$

$$x_1: C_1, \ldots, x_k: C_k \vdash^{p} e: C$$

## Theorem

*Let $\Gamma$ be a typing context and $\sigma$ a substitution consistent with $\Gamma$. Suppose $\sigma \overset{m}{\vdash} e \Rightarrow v$ and $\Gamma \overset{p(\Gamma)}{\vdash} e : C$; we obtain:*

$$\Phi(\sigma, \Gamma) + \log(p(\Gamma)) - \Phi(v) \geqslant m \ .$$

## Proof.

Let $\Xi$ denote the derivation of $\sigma \overset{m}{\vdash} e \Rightarrow v$ and $\Pi$ denote the proof of $\Gamma \overset{p(\Gamma)}{\vdash} e : C$. The theorem is proven by main induction on $|\Pi|$ and side induction on $|\Xi|$ ∎

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## Example

consider the function `splay`: $A \times T_n \to T_n$, defining splaying recursively, then `splay` is well-typed

## Strengths of Uniform Resource Analysis

- modularity and extensiblity
  - complexity problems
  - intermediary languages
- divide and conquer
- applications
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advice to students: don't listen to advice!

# Thank You for Your Attention!